

Minimizers of higher order gauge invariant functionals

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Abstract

We introduce higher order variants of the Yang-Mills functional that involve $(n-2)$ th order derivatives of the curvature. We prove coercivity and smoothness of critical points in Uhlenbeck gauge in dimensions $\dim M \leq 2n$. These results are then used to establish the existence of smooth minimizers on a given principal bundle $P \rightarrow M$ for subcritical dimensions $\dim M < 2n$. In the case of critical dimension $\dim M = 2n$ we construct a minimizer on a bundle which might differ from the prescribed one, but has the same Chern classes c_1, \dots, c_{n-1} . A key result is a removable singularity theorem for bundles carrying a $W^{n-1,2}$ -connection. This generalizes a recent result by Petrace and Rivière.

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1 Introduction

One of the limitations of elliptic Yang-Mills theory as a variational problem is that many of its features work only if the basis of the bundle has dimension ≤ 4 . Given a principal G -bundle $P \rightarrow M$ over a compact manifold M with compact structure group G , we consider G -connections $D_0 + A$ on it. (We will explain basic concepts of gauge theory in Section 2.1.) The squared curvature integral

$$YM(A) := \frac{1}{2} \int_M |F_A|^2 dx$$

is the *Yang-Mills functional*. Its variational theory is based on fundamental contributions by Uhlenbeck [Uh1] [Uh2] and Sedlacek [Se] from the 1980s. If $\dim M \leq 3$, there is a smooth minimizer of YM on any given bundle. If $\dim M = 4$, which is the critical and most natural case, then things are getting more interesting. A minimizing sequence $(A_j)_{j \in \mathbb{N}}$ for YM in a given bundle has to be looked at in a good gauge (roughly a good choice of bundle coordinates) in order to control Sobolev norms of A_j well enough to find a subsequence converging weakly to some

sort of minimizer A . The latter turns out to be smooth on most of M , but may initially have point singularities. Seeing those as possible singularities of the bundle itself, they can be removed to get a smooth minimizer A , but possibly on a different bundle, which is redefined even topologically in the singular points. The bundle in which to find the minimizer, however, is not arbitrary, since some of its topological invariants coincide with those of the bundle we started with.

It is the gauge choice for the minimizing sequence that may fail in dimensions $\dim M \geq 5$. This can be fixed partially, since the gauge theorem from [Uh2] works just as well on $2n$ -dimensional manifolds for

$$YM^n(A) := \frac{1}{n} \int_M |F_A|^n dx$$

instead of YM . Therefore, there may be hope that the variational approach we described for YM on 4-dimensional manifolds can be modified to work for YM^n on $2n$ -dimensional manifolds. Part of this program has been worked out, namely [Uh2] works for the gauge choice, and Isobe [Is] has worked out some local regularity theorem which can be applied to minimizers of YM^n . However, we think that there are two points where minimizing YM^n in $2n$ dimensions is not as good as minimizing YM in 4 dimensions.

- (1) Minimizers cannot be expected to be smooth, since the functional is degenerate. The best we can hope for, and that is essentially what has been proven in [Is], is $C^{1,\alpha}$ -regularity (except maybe for point singularities).
- (2) Uhlenbeck's singularity removal theorem from [Uh1] has been improved significantly by Petrasche and Rivière [PR] by removing the assumption that A is Yang-Mills. In fact, the existence of *any* $W^{1,2}$ -connection on a bundle over M minus one point implies that the bundle can be continued to give a bundle over all of M , provided that $\dim M \leq 4$. It is our impression that the arguments from [PR] do not carry over to $W^{1,n}$ -connections for $\dim M = 2n$.

The starting point for our paper is the following. *Both problems (1) and (2) do not occur if we work in $W^{n-1,2}$ instead of $W^{1,n}$ for $\dim M \leq 2n$.* As we shall prove in Theorem 5.2, the existence of a $W^{n-1,2}$ -connection on a $2n$ -dimensional bundle with one fibre missing implies that the singularity can be removed from the bundle. This directly generalizes the corresponding theorem from [PR] and therefore helps to get around problem (2). To handle the issue mentioned in (1), we have to work with *nondegenerate* functionals that control the $W^{n-1,2}$ -norm (and hence also the $W^{1,n}$ -norm). Therefore let us try to write down functionals that do the job and can then be used to replace YM^n .

A word on gauge invariance first, which is one of the basic features that make the Yang-Mills functional interesting. For any *gauge transformation*, i.e. any sufficiently regular equivariant map $u : P \rightarrow G$, we have $YM(A) = YM(u^*A)$, because F_A transforms like

$$F_{u^*A} = u^{-1} F_A u,$$

and u as well as u^{-1} act by isometries. One of the issues we have to deal with is finding *higher order* functionals that show the same gauge invariance. There has been some work in this direction, actually, at least for the $n = 3$ case. In [BU], Bejan and Urakawa have defined the Bi-Yang-Mills functional

$$YM_2(A) := \frac{1}{2} \int_M |d_A^* F_A|^2 dx,$$

where here d_A^* is the covariant exterior co-differential. It is gauge-invariant, since the Euler-Lagrange equation $d_A^* F_A = 0$ of YM must be gauge-invariant. In [IIU1] and [IIU2], some more of the basic properties of YM_2 are explored.

For our purpose of constructing minimizers, the functional YM_2 is not quite suitable, since it does not control any L^p -norm of $|F_A|$, which is bad news for controlling minimizing sequences. But we can add YM^3 to it, arriving at

$$Y_3(A) := \int_M (|d_A^* F_A|^2 + |F_A|^3) dx,$$

for which, as will follow from our results, all of the Uhlenbeck-Sedlacek program described above can be performed similarly in dimensions ≤ 6 . Philosophically, $Y_3(A)$ can control the $W^{1,2}$ -norm of F_A since it obviously controls $d_A^* F_A$ in L^2 , and $d_A F_A$ is always 0 by Bianchi's identity. Hodge theory says that DF_A is controlled once you can control dF_A and $d^* F_A$, so what we need is some "nonlinear variant" of Hodge theory. We cannot work with $|d^* F_A|$ directly, because it is not gauge invariant.

We go on constructing higher order gauge invariant functionals inductively. First derivatives $d_A^* F_A$ should be controlled by norms of $d_A d_A^* F_A$ and $d_A^* d_A^* F_A$, both of which have gauge invariant norms, see Section 2.2. But $d_A^* d_A^* F_A = - * [F_A, * F_A]$ happens to be of lower order, so $d_A d_A^* F_A$ alone should be enough to control first derivatives $d_A^* F_A$, and hence second derivatives of F_A . The functional

$$Y_4(A) := \int_M (|d_A d_A^* F_A|^2 + |F_A|^4) dx,$$

does the job, for $\dim M \leq 8$. Of course, we can iterate our considerations and find functionals suitable for our program. Abbreviating

$$d_A^{*\wedge n} := \begin{cases} (d_A d_A^*)^{n/2} & \text{if } n \text{ even,} \\ d_A^* (d_A d_A^*)^{(n-1)/2} & \text{if } n \text{ is odd,} \end{cases}$$

we define

$$Y_n(A) := \int_M (|d_A^{*\wedge n-2} F_A|^2 + |F_A|^n) dx.$$

These are gauge invariant, and scaling invariant if $\dim M = 2n$. Moreover, they will turn out to be coercive when put in the gauge found by Uhlenbeck [Uh2]. And, of course, being quadratic in the highest order, they are then nondegenerate, which opens the possibility of proving C^∞ for minimizers instead of $C^{1,\alpha}$, thus addressing

problem (1). It looks like we have found good candidates for functionals to look at.

From another point of view, the functionals Y_n may be not the best choice. They are no perturbations of the original Yang-Mills functional. We may wish to minimize “Yang-Mills plus something of higher order” and even think of that “something” being multiplied by some small $\varepsilon > 0$. The higher order terms we described so far do need the $|F_A|^n$ -term in order to be coercive. But if we are prepared to leave the realm of exterior forms, we can proceed. Instead of using exterior derivatives d_A and d_A^* , we can try to use other combinations of exterior partial derivatives. It turns out that the norm of the *full* covariant derivative $|D_A F_A|$ is also gauge invariant, and so are its iterates $|D_A^k F_A|$. Using these, we come up with a second sequence of functionals, this time “perturbations of YM”, which read

$$Z_n(A) := \int_M (|D_A^{n-2} F_A|^2 + |F_A|^2) dx.$$

Both sorts of functionals have their advantages, and it turns out that they can be estimated against each other and against the (squared) “nonlinear $W^{n-2,2}$ -norms” of F_A , which are built like the usual Sobolev norms, but using D_A instead of D . For the functionals Y_n and Z_n , we will prove the following results concerning the existence and regularity of minimizers.

Section 3. *Global “coercivity”* in the sense that Y_n and Z_n either control the nonlinear $W^{n-2,2}$ -norms of F_A mentioned above. This is not real coercivity since it works only with gauge invariant quantities and controls F_A instead of A itself. The exact statement is Theorem 3.1.

Section 4. *Local coercivity* in Uhlenbeck gauge in $\leq 2n$ dimensions. This is essential for extracting weak limits from minimizing sequences, and is performed in Theorem 4.1.

Section 5. *Removability of point singularities of bundles.* As remarked above, a point singularity of a bundle can be removed once we know the existence of a $W^{n-1,2}$ -connection around the missing point. Even better, we need only a connection for which Y_n or Z_n is finite. This key result helps us removing point singularities from the minimizing connection constructed below, which by construction is in $W^{n-1,2}$. See Theorem 5.2.

Section 6. Weak formulations of the *Euler-Lagrange equations*.

Section 7. *Smoothness of weak solutions* of the Euler-Lagrange equations, again for $\dim M \leq 2n$. This regularity result can be found in Theorem 7.1.

Section 8.1. *Existence of minimizers in the critical dimension $\dim M = 2n$.* As mentioned before, the choice of Uhlenbeck gauges – which is necessary to overcome the lack of coercivity of the functional – can be achieved uniformly for the minimizing sequence only away from finitely many points. This

results in finitely many singularities that might develop in the bundle during the minimizing process. However, the removable singularities theorem helps us to remove the singularities of the bundle, and then, using our regularity theorem, also the singularities of the minimizer. This minimizer, singularities having been removed, lives on a new bundle that might differ from the prescribed bundle. But it still has the same Chern classes $c_1(P), \dots, c_{n-1}(P)$ as the original bundle. For the detailed statement of the result, we refer to Theorem 8.3.

Section 8.2. *Existence of minimizers in subcritical dimensions* $\dim M < 2n$. In this case we can start with an arbitrary principal bundle with any compact structure group G and can construct a minimizer on the given bundle. Moreover, the constructed minimizing connection is smooth by the regularity theorem. The existence result is Theorem 8.4.

2 Basics

2.1 Basic facts on gauge theory

In this section we briefly recall those facts on connections on principal bundles that will be relevant for the present article. For a more thorough exposition of the theory, we refer to [We, App. A].

Throughout this paper, we fix a smooth compact Riemannian manifold M of dimension $m := \dim M \leq 2n$ and a compact Lie group G , the Lie algebra of which will be denoted by \mathfrak{g} . A *principal bundle* $\pi : P \rightarrow M$ over M with structure group G can be described by an open cover $\{U_\alpha\}_{\alpha=1}^L$ of M and local trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$. The trivializations give rise to transition functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$ defined by $\phi_{\alpha\beta}\phi_\beta = \phi_\alpha$ for all parameters $1 \leq \alpha, \beta \leq L$ with $U_\alpha \cap U_\beta \neq \emptyset$. From the definition of the transition functions, it is immediate to check the *cocycle conditions*

$$\phi_{\alpha\alpha} \equiv 1 \quad \text{and} \quad \phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma, \quad (2.1)$$

provided $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Conversely, any set of smooth functions $\{\phi_{\alpha\beta}\}_{\alpha,\beta}$ that satisfies the cocycle conditions (2.1) defines a principal G -bundle with transition functions $\phi_{\alpha\beta}$ relative to the open cover $\{U_\alpha\}$.

A *gauge transformation* on P is an equivariant smooth map $u : P \rightarrow G$. Using the trivializations ϕ_α of the bundle P , the gauge transformation can alternatively be characterized by its localizations $u_\alpha : U_\alpha \rightarrow G$, which are related by the transition identity

$$u_\beta = \phi_{\alpha\beta}^{-1} u_\alpha \phi_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta.$$

A smooth connection $D_0 + A$ on the principal bundle P is formally an element of the space $D_0 + C^\infty(M, T^*M \otimes \mathfrak{g}_P)$, where D_0 is a fixed smooth reference connection on P and \mathfrak{g}_P denotes the associated \mathfrak{g} -bundle, cf. [We, App. B]. See [We, App. A] for

the precise definition. For most of the present article however, it will be sufficient to think of A as of the entity of its localizations $(\phi_\alpha)_*A = A_\alpha \in C^\infty(U_\alpha, T^*M \otimes \mathfrak{g})$ subject to the trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ of the principal bundle, which are given by

$$A_\alpha(d_p\pi(v)) = \phi_\alpha(p)A(d_p\pi(v))\phi_\alpha^{-1} + d\phi_\alpha(v)\phi_\alpha^{-1}(p)$$

for all $p \in \pi^{-1}(U_\alpha)$ and $v \in T_pP$. In fact, the set $\{A_\alpha\}_{\alpha=1}^L$ contains the same information as A . The localizations A_α , $1 \leq \alpha \leq L$ are related by the identity

$$A_\beta = \phi_{\alpha\beta}^{-1}A_\alpha\phi_{\alpha\beta} + \phi_{\alpha\beta}^{-1}d\phi_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta \quad (2.2)$$

for all parameters α, β for which the latter set is non-empty. On the other hand, if we are given a set of smooth local connections $A_\alpha \in C^\infty(U_\alpha, T^*M \otimes \mathfrak{g})$, $\alpha = 1, \dots, L$ that satisfy the compatibility conditions (2.2) with respect to the transition functions $\phi_{\alpha\beta}$ of the bundle P , we can find a smooth connection A on P with trivializations $(\phi_\alpha)_*A = A_\alpha$.

The gauge transformation u acts on the connection A via

$$u^*A := u^{-1}Au + u^{-1}du.$$

The *curvature* of a connection A is given by

$$F_A := dA + A \wedge A.$$

It is well-known that the curvature is gauge-equivariant in the sense that for every gauge transformation $u : P \rightarrow G$ there holds

$$F_{u^*A} = u^{-1}F_A u.$$

Finally, we note that Sobolev spaces of connections can be defined by

$$\mathcal{A}^{k,p}(P) := D_0 + W^{k,p}(M, T^*M \otimes \mathfrak{g}_P),$$

where D_0 is a fixed smooth reference connection on P and \mathfrak{g}_P denotes the associated \mathfrak{g} -bundle, cf. [We, App. B]. Locally, $W^{n-1,2}$ -connections on U_α are represented by $A_\alpha \in W^{n-1,2}(U_\alpha, T^*M \otimes \mathfrak{g})$. Accordingly, we will consider local gauge transformations of class $W^{n,2}$, in other words, maps $u_\alpha \in W^{n,2}(U_\alpha, G)$.

2.2 Calculations with differential forms

We now consider \mathfrak{g} -valued differential forms on a coordinate chart $U \subset \mathbb{R}^m$. For a \mathfrak{g} -valued k -form A and a \mathfrak{g} -valued ℓ -form B , we introduce the abbreviation

$$[A, B] := A \wedge B - (-1)^{k\ell} B \wedge A.$$

With this notation, we define in the case $k = 1$

$$d_A B = dB + [A, B] \quad \text{and} \quad d_A^* B := d^* B + (-1)^{m+1} [A, *B].$$

We introduce the following notations for higher order exterior derivatives of differential forms. For 1-forms $C \in W^{k,1}(U, \wedge^1 \mathbb{R}^m \otimes \mathfrak{g})$, respectively 2-forms $B \in W^{k,1}(U, \wedge^2 \mathbb{R}^m \otimes \mathfrak{g})$, where $k \in \mathbb{N}$, we use the notations

$$d^{\wedge k} C := \begin{cases} \underbrace{dd^* \cdots d}_k C, & k \text{ odd}, \\ \underbrace{d^* \cdots d}_k C, & k \text{ even}, \end{cases} \quad d^{*\wedge k} B := \begin{cases} \underbrace{d^* d \cdots d^*}_k B, & k \text{ odd}, \\ \underbrace{d \cdots d^*}_k B, & k \text{ even}. \end{cases}$$

Similarly, for a connection A we introduce nonlinear versions

$$d_A^{\wedge k} C := \begin{cases} \underbrace{d_A d_A^* \cdots d_A}_k C, & k \text{ odd}, \\ \underbrace{d_A^* \cdots d_A}_k C, & k \text{ even}, \end{cases} \quad d_A^{*\wedge k} B := \begin{cases} \underbrace{d_A^* d_A \cdots d_A^*}_k B, & k \text{ odd}, \\ \underbrace{d_A \cdots d_A^*}_k B, & k \text{ even}. \end{cases}$$

We apply this in particular to $B = F_A$ to define higher order exterior derivatives $d_A^{*\wedge k} F_A$ of the curvature.

Furthermore, we define higher order total derivatives $D_A^k F_A$ by exploiting the fact that the connection A induces covariant derivatives on vector bundles associated with the principal bundle P . More precisely, $D_A^k F_A$ is a section of the bundle $\otimes^k T^* M \otimes \wedge^2 T^* M \otimes \mathfrak{g}_P$ and is defined inductively by

$$D_A^k F_A := D_A(D_A^{k-1} F_A) \quad \text{for } k \in \mathbb{N}.$$

Here, the first D_A on the right-hand side denotes the covariant derivative on $\otimes^{k-1} T^* M \otimes \wedge^2 T^* M \otimes \mathfrak{g}_P$ that is induced by the Levi-Civita connection on TM and the connection A on P .

By D_A^* we denote the formal adjoint of D_A . For later reference, we remark the existence of constants depending on the bundle, such that

$$|d_A B| \leq C |D_A B|, \quad |d_A^* B| + |D_A^* B| \leq C(|D_A B| + |B|) \quad (2.3)$$

for all forms B in the above bundle.

The above definitions of derivatives of F_A provide us with two classes of higher order functionals, namely

$$Y_n(A) := \int_M (|d_A^{*\wedge n-2} F_A|^2 + |F_A|^n) dx,$$

and

$$Z_n(A) := \int_M (|D_A^{n-2} F_A|^2 + |F_A|^2) dx.$$

For our purposes, it is crucial that both types of functionals are gauge invariant. This is a consequence of the following three lemmas.

Lemma 2.1 *Let B_A be some \mathfrak{g} -valued 2-form that transforms according to*

$$B_{u^*A} = u^{-1}B_A u.$$

Then we also have

$$d_{u^*A}^* B_{u^*A} = u^{-1} d_A^* B_A u.$$

Proof. We compute, using the gauge equivariance of B_A and $0 = d(uu^{-1}) = du u^{-1} + u d(u^{-1})$,

$$\begin{aligned} *d * B_{u^*A} &= *d *(u^{-1}B_A u) \\ &= *(u^{-1}d(*B_A)u + d(u^{-1}) \wedge (*B_A)u + (-1)^{m-2}u^{-1}(*B_A) \wedge du) \\ &= u^{-1}*d(*B_A)u - *(u^{-1}du \wedge u^{-1}(*B_A)u - (-1)^{m-2}u^{-1}(*B_A)u \wedge u^{-1}du) \\ &= u^{-1}*d * B_A u - *[u^{-1}du, *(u^{-1}B_A u)] \end{aligned}$$

and

$$\begin{aligned} *[u^*A, *B_{u^*A}] &= *[u^{-1}Au + u^{-1}du, *(u^{-1}B_A u)] \\ &= u^{-1}*[A, *B_A]u + *[u^{-1}du, *(u^{-1}B_A u)]. \end{aligned}$$

Thus we have

$$d_{u^*A}^* B_{u^*A} = -(-1)^m u^{-1}(*d * B_A + *[A, *B_A])u = u^{-1}d_A^* B_A u$$

as claimed. \square

Lemma 2.2 *Let C_A be some \mathfrak{g} -valued 1-form that transforms according to*

$$C_{u^*A} = u^{-1}C_A u.$$

Then we also have

$$d_{u^*A} C_{u^*A} = u^{-1}d_A C_A u.$$

Proof. As above,

$$\begin{aligned} dC_{u^*A} &= d(u^{-1}C_A u) \\ &= u^{-1}dC_A u + d(u^{-1}) \wedge C_A u - u^{-1}C_A \wedge du \\ &= u^{-1}dC_A u - u^{-1}du \wedge u^{-1}C_A u - u^{-1}C_A u \wedge u^{-1}du \\ &= u^{-1}dC_A u - [u^{-1}du, u^{-1}C_A u], \end{aligned}$$

as well as

$$\begin{aligned} [u^*A, u^{-1}C_A u] &= [u^{-1}Au + u^{-1}du, u^{-1}C_A u] \\ &= u^{-1}[A, C_A]u + [u^{-1}du, u^{-1}C_A u]. \end{aligned}$$

This implies

$$d_{u^*A}C_{u^*A} = dC_{u^*A} + [u^*A, u^{-1}C_Au] = u^{-1}d_AC_Au$$

as desired. \square

Lemma 2.3 *Let C_A be some \mathfrak{g} -valued multilinear form that transforms according to*

$$C_{u^*A} = u^{-1}C_Au.$$

Then we also have

$$D_{u^*A}C_{u^*A} = u^{-1}D_AC_Au.$$

We omit the proof because it is almost literally the same as the preceding one if one replaces d by D . \square

Since the curvature of a connection A transforms like $F_{u^*A} = u^{-1}F_Au$, the three preceding lemmas yield the

Corollary 2.4 *For the curvature F_A of a local connection A of class $W^{n-1,2}$ and a $W^{2,n}$ -gauge transformation u we have*

$$d_{u^*A}^{*\wedge(n-2)}F_{u^*A} = u^{-1}(d_A^{*\wedge(n-2)}F_A)u$$

and

$$D_{u^*A}^{n-2}F_{u^*A} = u^{-1}(D_A^{n-2}F_A)u.$$

*In particular, this implies the gauge invariance of the functionals Y_n and Z_n in the form $Y_n(u^*A) = Y_n(A)$, respectively $Z_n(u^*A) = Z_n(A)$.*

Remark 2.5 The reader may have expected $W^{n,2}$ gauge transformations instead of $W^{2,n}$ in the corollary. However, by the preceding lemmas, we see that $d_{u^*A}^{*\wedge(n-2)}F_{u^*A}$ and $D_{u^*A}^{n-2}F_{u^*A}$ are defined even if u is only in $W^{2,n}$. This is seen by iterating arguments like $D_{u^*A}F_{u^*A} = D_{u^*A}(u^{-1}F_Au)$, where only one derivative of u is needed on the right-hand side.

We have remarked in the introduction that d_Ad_A and $d_A^*d_A^*$, even when applied to forms like A or F_A , are differential operators of order 0. More precisely, we have

Lemma 2.6 *For all \mathfrak{g} -valued 2-forms B and \mathfrak{g} -valued 1-forms C , we have the identities*

$$\begin{aligned} d_A^*d_A^*B &= -*[F_A, *B], \\ d_Ad_AC &= [F_A, C]. \end{aligned}$$

Proof. The second assertion is more or less the definition of F_A , and moreover a simpler variant of the proof of the first assertion, which we now give.

We use $* * = (-1)^{k(m+1)}$ and $d^* = (-1)^{(k+1)m+1} * d *$ when operating on k -forms, and $[X, Y] = (-1)^{k\ell+1} [Y, X]$ when X is a k -form and Y is an ℓ -form. Therefore

$$\begin{aligned}
d_A^* d_A^* B &= -(-1)^m d_A^* (* d * B + * [A, * B]) \\
&= (-1)^m \left(* d * * d * B + * d * * [A, * B] \right. \\
&\quad \left. + * [A, * * d * B] + * [A, * * [A, * B]] \right) \\
&= - * \left(dd * B + d[A, * B] + [A, d * B] + [A, [A, * B]] \right) \\
&= - * \left([dA, * B] - [A, d * B] + [A, d * B] + [A, [A, * B]] \right) \\
&= - * \left([dA, * B] + [A, [A, * B]] \right).
\end{aligned}$$

The Jacobi identity (with correct signs) yields

$$[A, [A, * B]] + (-1)^{m-1} [A, [* B, A]] + [* B, [A, A]] = 0,$$

from which we infer

$$2[A, [A, * B]] = -[* B, [A, A]] = [[A, A], * B].$$

We insert this in our previous calculation to find

$$d_A^* d_A^* B = - * \left([dA, * B] + \frac{1}{2} [[A, A], * B] \right) = - * [F_A, * B]$$

as asserted. □

2.3 Gagliardo-Nirenberg interpolation

In order to deal with lower order derivatives, we rely on the Gagliardo-Nirenberg interpolation inequality in the following form (see [Ni, Thm. 1]).

Theorem 2.7 *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with the cone property and $u \in W^{k,r}(\Omega) \cap L^q(\Omega)$, where $1 \leq p, q \leq \infty$ and $k \in \mathbb{N}$. Then we have the inequality*

$$\|D^j u\|_{L^p(\Omega)} \leq C \|D^k u\|_{L^r(\Omega)}^{j/k} \|u\|_{L^q(\Omega)}^{1-j/k} + C \|u\|_{L^q(\Omega)}$$

provided $0 \leq j < k$ and $\frac{1}{p} = \frac{j}{k} \frac{1}{r} + (1 - \frac{j}{k}) \frac{1}{q}$. Here, the constant C depends only on Ω, k, j, q , and r .

3 Gauge invariant interpolation and Sobolev inequalities

We define the Sobolev spaces $W_A^{k,p}(M, \wedge^2(T^*M) \otimes \mathfrak{g}_P)$ containing all sections B of the bundle for which

$$\|B\|_{W_A^{k,p}(M)} := \sum_{j=0}^k \|D_A^j B\|_{L^p(M)}$$

is finite. The main purpose of this section is to verify that both $Y_n(A)$ and $Z_n(A)$ can control $\|F_A\|_{W_A^{n-2,2}(M)}$.

Theorem 3.1 (i) *For any smooth section A of $T^*M \otimes \mathfrak{g}_P$ for which $Y_n(A)$ is finite, we have $F_A \in W_A^{n-2,2}(M, \wedge^2(T^*M) \otimes \mathfrak{g}_P)$, and even*

$$\sum_{k=0}^{n-2} \|D_A^k F_A\|_{L^{2n/(k+2)}}^{2n/(k+2)} \leq C(Y_n(A) + Y_n(A)^{2/n}).$$

(ii) *Assume that $m = \dim M \leq 2n$. For any smooth section A of $T^*M \otimes \mathfrak{g}_P$ for which $Z_n(A)$ is finite, we have $F_A \in W_A^{n-2,2}(M, \wedge^2(T^*M) \otimes \mathfrak{g}_P)$, and even*

$$\sum_{k=0}^{n-2} \|D_A^k F_A\|_{L^{2n/(k+2)}}^{2n/(k+2)} \leq C(Z_n(A) + Z_n(A)^{n/2}).$$

In both cases, the constant C depends on M and n only.

Proof of (i). We remind the reader of the Weitzenböck formula, which somewhat symbolically reads

$$(d_A^* d_A + d_A d_A^* - D_A^* D_A)B = F_A \# B + R_M \# B,$$

where $F_A \# B$ and $R_M \# B$ mean universal bilinear forms applied to F_A or R_M , and B . Here R_M is the Riemannian curvature of M . Applying this to $B = F_A$ and multiplying with F_A , we find, using also Bianchi's identity $d_A F_A = 0$,

$$\begin{aligned} \|D_A F_A\|_{L^2}^2 &= \int_M \langle F_A, D_A^* D_A F_A \rangle dx \\ &= \int_M \langle F_A, d_A d_A^* F_A + F_A \# F_A + R_M \# F_A \rangle dx \\ &\leq \|d_A^* F_A\|_{L^2}^2 + \|F_A \# F_A \# F_A + R_M \# F_A \# F_A\|_{L^1}, \end{aligned}$$

which we write as

$$\|D_A F_A\|_{L^2}^2 \leq \|d_A^* F_A\|_{L^2}^2 + C\|F_A \# F_A \# F_A + F_A \# F_A\|_{L^1},$$

since R_M is a given smooth form. Now we proceed doing a similar calculation for $D_A^2 F_A$. We use, in that order (a) Weitzenböck with $B = D_A F_A$, (b) $d_A D_A B - D_A d_A B = F_A \# B$, $d_A^* D_A B - D_A d_A^* B = F_A \# B$ and $d_A F_A = 0$ by Bianchi, (c) Weitzenböck with $B = d_A^* F_A$, (d) $d_A^* d_A^* B = F_A \# B$ using Lemma 2.6 and the estimate (2.3). Those give

$$\begin{aligned}
& \|D_A^2 F_A\|_{L^2}^2 \\
& \leq \|d_A^* D_A F_A\|_{L^2}^2 + \|d_A D_A F_A\|_{L^2}^2 + C\|D_A F_A \# D_A F_A \# F_A + D_A F_A \# D_A F_A\|_{L^1} \\
& \leq C\|D_A d_A^* F_A\|_{L^2}^2 + C\|F_A \# F_A\|_{L^2}^2 + C\|D_A F_A \# D_A F_A \# F_A + D_A F_A \# D_A F_A\|_{L^1} \\
& \leq C\|d_A d_A^* F_A\|_{L^2}^2 + C\|d_A^* d_A^* F_A\|_{L^2}^2 + C\|D_A F_A \# D_A F_A \# F_A + D_A F_A \# D_A F_A\|_{L^1} \\
& \quad + C\|F_A \# F_A\|_{L^2}^2 + C\|d_A^* F_A \# d_A^* F_A \# F_A + d_A^* F_A \# d_A^* F_A\|_{L^1} \\
& \leq C\|d_A d_A^* F_A\|_{L^2}^2 + C\|LOT\|_{L^1}
\end{aligned}$$

where the lower order terms LOT are of the types

$$\begin{aligned}
& F_A \# F_A \# F_A \# F_A, \quad F_A \# F_A \# F_A, \quad F_A \# F_A, \\
& D_A F_A \# D_A F_A \# F_A, \quad D_A F_A \# D_A F_A, \quad D_A F_A \# F_A \# F_A, \quad D_A F_A \# F_A.
\end{aligned}$$

Iterating this by a long but straightforward induction argument, we find

$$\|D_A^{k-2} F_A\|_{L^2}^2 \leq C\|d_A^{* \wedge k-2} F_A\|_{L^2}^2 + C\|LOT\|_{L^1}$$

for every $k \in \{2, \dots, n\}$. Here the lower order terms each are of the form

$$\#_{j=0}^{k-3} (D_A^j F_A)^{\# h_j} =: TLOT$$

(meaning “this lower order term”) with

$$4 \leq s := \sum_{j=0}^{k-3} (j+2)h_j \leq 2k, \quad \sum_{j=0}^{k-3} h_j \geq 2.$$

Now we distinguish three cases.

Case 1. Only one of the h_j is $\neq 0$. Then $2 \leq h_j \leq \frac{2k}{j+2}$ and it is elementary to estimate

$$\|TLOT\|_{L^1} \leq C\|D_A^j F_A\|_{L^{h_j}}^{h_j} \leq C\left(\|D_A^j F_A\|_{L^2}^2 + \|D_A^j F_A\|_{L^{2k/(j+2)}}^{2k/(j+2)}\right).$$

Case 2. For all j with $h_j \neq 0$, we have $j+2 \leq \frac{s}{2}$. Then we use Young’s inequality with exponents $\frac{s}{(j+2)h_j}$ whenever $h_j \neq 0$, and find, using $s \leq 2k$ and $\frac{s}{j+2} \geq 2$,

$$\|TLOT\|_{L^1} \leq C \sum_{j=0}^{k-3} \|D_A^j F_A\|_{L^{s/(j+2)}}^{s/(j+2)} \leq C \sum_{j=0}^{k-3} \left(\|D_A^j F_A\|_{L^2}^2 + \|D_A^j F_A\|_{L^{2k/(j+2)}}^{2k/(j+2)} \right).$$

Case 3. The largest j with $h_j \neq 0$, let us call it J , satisfies $J + 2 > \frac{s}{2}$. Then $j + 2 \leq \frac{s}{2}$ for all other terms with $h_j \neq 0$, and we must have $h_J = 1$. We use Young's inequality with exponents 2 for the J -term, and $\frac{2(s-J-2)}{(j+2)h_j}$ for the others. The exponents $\frac{2(s-J-2)}{j+2}$ that occur in the following calculation are ≥ 2 because $j + 2 \leq s - J - 2$, and they are $\leq \frac{2k}{j+2}$ because $s - J - 2 < s - \frac{s}{2} \leq k$. This justifies the estimate

$$\begin{aligned} \|TLOT\|_{L^1} &\leq C \|D_A^J F_A\|_{L^2}^2 + C \sum_{j < J, h_j \neq 0} \|D_A^j F_A\|_{L^{2(s-J-2)/(j+2)}}^{2(s-J-2)/(j+2)} \\ &\leq C \sum_{j=0}^{k-3} \left(\|D_A^j F_A\|_{L^2}^2 + \|D_A^j F_A\|_{L^{2k/(j+2)}}^{2k/(j+2)} \right). \end{aligned}$$

Summing up over all lower order terms therefore gives

$$\|D_A^{k-2} F_A\|_{L^2}^2 \leq \|d_A^{*\wedge k-2} F_A\|_{L^2}^2 + C \sum_{j=0}^{k-3} \left(\|D_A^j F_A\|_{L^2}^2 + \|D_A^j F_A\|_{L^{2k/(j+2)}}^{2k/(j+2)} \right). \quad (3.1)$$

Now we start our interpolation considerations by remarking that

$$\|d_A^{*\wedge k} F_A\|_{L^2} \leq \varepsilon \|d_A^{*\wedge n-2} F_A\|_{L^2} + C(\varepsilon) \|F_A\|_{L^2} \quad (3.2)$$

for $k \in \{0, \dots, n-3\}$ is straightforward. Similarly,

$$\|D_A^k F_A\|_{L^2} \leq \varepsilon \|D_A^{n-2} F_A\|_{L^2} + C(\varepsilon) \|F_A\|_{L^2}, \quad (3.3)$$

which is still easy to prove, but slightly more subtle. This is in fact an easier variant of the following one. It reads

$$\begin{aligned} \|D_A^k F_A\|_{L^{2n/(k+2)}}^{2n/(k+2)} &= \int_M \langle D_A^k F_A, |D_A^k F_A|^{\frac{2n}{k+2}-2} D_A^k F_A \rangle dx \\ &= \int_M \langle D_A^{k-1} F_A, |D_A^k F_A|^{\frac{2n}{k+2}-2} D_A^* D_A^k F_A \rangle dx \\ &\quad + C \int_M \left\langle D_A^{k-1} F_A, |D_A^k F_A|^{\frac{2n}{k+2}-4} \langle \langle D_A^{k+1} F_A, D_A^k F_A \rangle, D_A^k F_A \rangle \right\rangle dx \\ &\leq C(\varepsilon) \|D_A^{k-1} F_A\|_{L^{2n/(k+1)}}^{2n/(k+1)} + \varepsilon \|D_A^{k+1} F_A\|_{L^{2n/(k+3)}}^{2n/(k+3)} \\ &\quad + \varepsilon \left(\|D_A^k F_A\|_{L^{2n/(k+2)}}^{2n/(k+2)} + \|D_A^k F_A\|_{L^{2n/(k+3)}}^{2n/(k+3)} \right) \\ &\leq C(\varepsilon) \|D_A^{k-1} F_A\|_{L^{2n/(k+1)}}^{2n/(k+1)} + \varepsilon \|D_A^{k+1} F_A\|_{L^{2n/(k+3)}}^{2n/(k+3)} \\ &\quad + \varepsilon \left(\|D_A^k F_A\|_{L^{2n/(k+2)}}^{2n/(k+2)} + \|D_A^k F_A\|_{L^2}^2 \right) \end{aligned}$$

At the second “=”, we have used that D_A is a metric connection. At exactly that point, the version of it using $d_A^{*\wedge k}$ instead of D_A^k would fail, since we would get

one D_A , anyway. In some sense, this is why the proof of (i) is not straightforward. For the first “ \leq ”, we have used Young’s inequality and (2.3).

Absorbing the second-last term into the left-hand side and estimating the last one with (3.3), we iterate and find

$$\|D_A^k F_A\|_{L^{2n/(k+2)}}^{2n/(k+2)} \leq \varepsilon \|D_A^{n-2} F_A\|_{L^2}^2 + C(\varepsilon)(\|F_A\|_{L^2}^2 + \|F_A\|_{L^n}^n) \quad (3.4)$$

for all $k \in \{0, \dots, n-3\}$. Using (3.4), (3.1), (3.4) again, and then (3.3), we have

$$\begin{aligned} \sum_{k=0}^{n-2} \|D_A^k F_A\|_{L^{2n/(k+2)}}^{2n/(k+2)} &\leq C(\|D_A^{n-2} F_A\|_{L^2}^2 + \|F_A\|_{L^2}^2) + C\|F_A\|_{L^n}^n \\ &\leq C\left(\|d_A^{*\wedge n-2} F_A\|_{L^2}^2 + \sum_{k=0}^{n-3} (\|D_A^k F_A\|_{L^2}^2 + \|D_A^k F_A\|_{L^{2n/(k+2)}}^{2n/(k+2)})\right) \\ &\leq C(\varepsilon)\left(\|d_A^{*\wedge n-2} F_A\|_{L^2}^2 + \sum_{k=0}^{n-3} \|D_A^k F_A\|_{L^2}^2 + \|F_A\|_{L^n}^n\right) + \varepsilon \|D_A^{n-2} F_A\|_{L^2}^2 \\ &\leq C(\varepsilon)(\|d_A^{*\wedge n-2} F_A\|_{L^2}^2 + \|F_A\|_{L^2}^2 + \|F_A\|_{L^n}^n) + C\varepsilon \|D_A^{n-2} F_A\|_{L^2}^2. \end{aligned}$$

Absorbing the last term, we have proven assertion (i).

Proof of (ii). This is easier, and interesting in its own right. It is well-known that Sobolev inequalities for D_A hold with constants not depending on A . This is because

$$2|B||D|B| = |D|B|^2| = |D\langle B, B \rangle| = 2|\langle B, D_A B \rangle| \leq 2|B||D_A B|$$

implies $|D|B| \leq |D_A B|$ for all A , and hence

$$\|B\|_{L^{p^*}} = \||B|\|_{L^{p^*}} \leq C_S \|D|B|\|_{W^{1,p}} \leq C_S \|D_A B\|_{W^{1,p}}.$$

Iterating, we also have higher order Sobolev inequalities with constants independent of A , and in particular, using $\dim M \leq 2n$,

$$\begin{aligned} \sum_{k=0}^{n-2} \|D_A^k F_A\|_{L^{2n/(k+2)}} &\leq C\|F_A\|_{W_A^{n-2,2}} \leq C(\|D_A^{n-2} F_A\|_{L^2} + \|F_A\|_{L^2}) \\ &\leq CZ_n(A)^{1/2}, \end{aligned}$$

where the second “ \leq ” comes from (3.3). This completes the proof of Theorem 3.1. \square

Remark 3.2 *On manifolds with boundary, in particular on balls in M , we get similar assertions. We have to use cutoff functions near the boundary, however, and therefore get only $F_A \in W_A^{n-2,2}$ locally, with estimates on every compact subset away from the boundary.*

4 Uhlenbeck type estimates

In dimension $m = 2n$, Uhlenbeck [Uh2] showed that smallness of $\|F_A\|_{L^n(B_r)}$ ensures that by a suitable gauge change, we can achieve $d^*A = 0$, $*A|_{\partial B_r} = 0$ and

$$\|A\|_{L^{2n}(B_r)} + \|DA\|_{L^n(B_r)} \leq c\|F_A\|_{L^n(B_r)}.$$

We will need something similar for higher order.

Theorem 4.1 (higher order Uhlenbeck estimates) *Assume $n \geq 2$, $m \leq 2n$, and let $B_r = B_r(a)$ be any ball of radius $r \in (0, 1)$ in M . There is a constant $\kappa = \kappa(M) \in (0, 1)$ such that the following holds. Assume that $A \in W^{1,n}(B_r)$ satisfies $A \in W_{\text{loc}}^{n-1,2}(B_r \setminus \{0\})$ and*

$$D_A^\ell F_A \in L^{2n/(\ell+2)}(B_r) \quad \text{for } \ell = 1, \dots, n-2 \quad (4.1)$$

(which is in particular satisfied in the case $A \in W^{n-1,2}(B_r)$). Moreover, we assume that the curvature is small in the sense

$$r^{\frac{2n-m}{n}} \|F_A\|_{L^n(B_r)} < \kappa. \quad (4.2)$$

Then the Uhlenbeck gauged version Ω of A obeys the estimate

$$\begin{aligned} & \sum_{\ell=0}^{n-1} r^{\frac{2n-m}{2n}(\ell+1)} \|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_{r/2})} \\ & \leq Cr^{\frac{2n-m}{2}} \|D_\Omega^{n-2} F_\Omega\|_{L^2(B_r)} + Cr^{\frac{2n-m}{n}} \|F_\Omega\|_{L^n(B_r)} \end{aligned} \quad (4.3)$$

with a constant C depending on M only. (Note that the powers of r all disappear in the critical dimension $m = 2n$.)

Remark 4.2 The assumption $A \in W_{\text{loc}}^{n-1,2}(B_r \setminus \{0\})$ is technical. It could be replaced by any weaker assumption that ensures all terms in (4.1) to be well-defined a.e.

Proof of the theorem. By scaling invariance, we may restrict ourselves to the case $r = 1$. We trivialize the bundle over B_1 , and we consider B_1 to be the Euclidean ball $B_1 = B_1(0) \subset \mathbb{R}^m$, equipped with some Riemannian metric γ . All constants involving γ can be chosen independent of the choice of the ball, because M is compact.

We will later choose $\kappa > 0$ not larger than in Uhlenbeck's gauge Theorem [Uh2]. Then our assumption $\|F_A\|_{L^n} < \kappa$ allows us to find a gauge $u \in W^{2,n}(B_1, G)$ such that $\Omega := u^*A \in W^{1,n}(B_1)$ satisfies $d^*\Omega = 0$ and moreover,

$$\|\Omega\|_{L^{2n}(B_1)} + \|D\Omega\|_{L^n(B_1)} \leq C\|F_A\|_{L^n(B_1)} < C\kappa. \quad (4.4)$$

From gauge invariance of the total derivatives $D_A^\ell F_A$ and (4.1) for $r = 1$, we know

$$\|D_\Omega^\ell F_\Omega\|_{L^{2n/(\ell+2)}(B_1)} = \|D_A^\ell F_A\|_{L^{2n/(\ell+2)}(B_1)} < \infty$$

for $\ell = 0, \dots, n-2$. The remainder of the proof is divided into three steps.

Step 1: Controlling $D^k d\Omega$ by $D_\Omega^k F_\Omega$ for $k = 1, \dots, n-2$.

Writing D for the total derivative, applied separately to the coefficients of F_Ω , we have a relation of the form

$$\begin{aligned} DF_\Omega &= D_\Omega F_\Omega + F_\Omega \# \Omega + F_\Omega \# \Gamma \\ &= D_\Omega F_\Omega + D\Omega \# \Omega + \Omega \# \Omega \# \Omega + D\Omega \# \Gamma + \Omega \# \Omega \# \Gamma, \end{aligned}$$

where Γ represents the Christoffel symbols of the manifold M . Keeping in mind that Γ and all its derivatives are bounded by constants depending only on M , we can generalize the last formula inductively to higher order, with the result

$$D^k F_\Omega = D_\Omega^k F_\Omega + \sum_{j \in J_k} \#_{i=1}^{\ell} D^{j_i-1} \Omega, \quad (4.5)$$

for $k = 1, \dots, n-2$, where we abbreviated

$$J_k := \{j = (j_1, \dots, j_\ell) : \ell \geq 1, 1 \leq j_i \leq k+1, 2 \leq j_1 + \dots + j_\ell \leq k+2\}.$$

Applying first Hölder's inequality with exponents $(k+2)/j_i$ and then Young's inequality with exponents $(j_1 + \dots + j_\ell)/j_i$, we infer

$$\begin{aligned} &\|D^k F_\Omega\|_{L^{2n/(k+2)}(B_\rho)} \\ &\leq C \|D_\Omega^k F_\Omega\|_{L^{2n/(k+2)}(B_\rho)} + C \sum_{j \in J_k} \prod_{i=1}^{\ell} \|D^{j_i-1} \Omega\|_{L^{2n/j_i}(B_\rho)} \\ &\leq C \|D_\Omega^k F_\Omega\|_{L^{2n/(k+2)}(B_\rho)} + C \sum_{\ell=0}^k \left(\|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_\rho)} + \|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_\rho)}^{(k+2)/(\ell+1)} \right), \end{aligned}$$

for every $\rho \in [\frac{1}{2}, 1]$. From Leibnitz' rule and Young's inequality, we deduce

$$\|D^k(\Omega \wedge \Omega)\|_{L^{2n/(k+2)}(B_\rho)} \leq C \sum_{\ell=0}^k \|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_\rho)}^{(k+2)/(\ell+1)}.$$

Joining the two preceding estimates and keeping in mind $F_\Omega = d\Omega + \Omega \wedge \Omega$, we arrive at

$$\begin{aligned} &\|D^k d\Omega\|_{L^{2n/(k+2)}(B_\rho)} \\ &\leq C \|D_\Omega^k F_\Omega\|_{L^{2n/(k+2)}(B_\rho)} + C \sum_{\ell=0}^k \left(\|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_\rho)} + \|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_\rho)}^{(k+2)/(\ell+1)} \right) \end{aligned} \quad (4.6)$$

for $k = 1, \dots, n-2$.

Step 2: Proof of $\Omega \in W_{\text{loc}}^{n-1,2}(B_1)$.

We will prove by induction over $k = 1, \dots, n-1$ that

$$D^\ell \Omega \in L_{\text{loc}}^{2n/(\ell+1)}(B_1) \quad \text{for } \ell = 0, \dots, k. \quad (4.7)$$

For $k = 1$, this is a consequence of Uhlenbeck's result (4.4). Next, we assume that (4.7) is valid for $k \in \{1, \dots, n-2\}$ and wish to prove it for $k+1$. To this end, we calculate

$$-\Delta \Omega = d^* d \Omega + d d^* \Omega = d^* d \Omega,$$

which implies

$$\begin{aligned} \|\Delta D^{k-1} \Omega\|_{L^{2n/(k+2)}(B_\rho)} &\leq C \|d \Omega\|_{W^{k, 2n/(k+2)}(B_\rho)} \\ &\leq C \|D^k d \Omega\|_{L^{2n/(k+2)}(B_\rho)} + C \|\Omega\|_{W^{k, 2n/(k+2)}(B_\rho)} < \infty \end{aligned}$$

for every $\rho \in [\frac{1}{2}, 1]$. The finiteness of the right-hand side is a consequence of (4.6) and the induction assumption (4.7). Now classical Calderón-Zygmund theory implies $D^{k+1} \Omega \in L_{\text{loc}}^{2n/(k+2)}(B_1)$. Proceeding in this manner up to the order $k = n-1$, we arrive at $D^{n-1} \Omega \in L_{\text{loc}}^2(B_1)$, which establishes the claim $\Omega \in W_{\text{loc}}^{n-1,2}(B_1)$.

Step 3: Proof of estimate (4.3).

The Sobolev regularity established in the preceding step now justifies the following calculations that will lead to the desired estimate. For given radii R, S with $\frac{1}{2} \leq R < S \leq \frac{3}{4}$, we choose a cut-off function $\varphi \in C_{\text{cpt}}^\infty(B_S, [0, 1])$ with $\varphi \equiv 1$ on B_R and $\|D^j \varphi\|_{L^\infty} \leq C/(S-R)^j$ for all $j \in \{1, \dots, n-1\}$. Using once more $d^* \Omega = 0$, we obtain

$$\Delta(\varphi \Omega) = d^*(d\varphi \wedge \Omega + \varphi d\Omega) + d(d\varphi \cdot \Omega).$$

Differentiating this identity $(n-3)$ times, using the properties of φ and $S-R < 1$, we deduce

$$\|\Delta D^{n-3}(\varphi \Omega)\|_{L^2} \leq C \|D^{n-2} d\Omega\|_{L^2(B_S)} + C \sum_{\ell=0}^{n-2} \frac{1}{(S-R)^{n-\ell-1}} \|D^\ell \Omega\|_{L^2(B_S)}.$$

Next, we apply (4.6) for $k = n-2$ and $\rho = S$, with the result

$$\begin{aligned} \|\Delta D^{n-3}(\varphi \Omega)\|_{L^2} &\leq C \|D_\Omega^{n-2} F_\Omega\|_{L^2(B_S)} \\ &\quad + C \sum_{\ell=0}^{n-2} \left(\frac{1}{(S-R)^{n-\ell-1}} \|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_S)} + \|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_S)}^{n/(\ell+1)} \right). \end{aligned}$$

At this stage, we once more apply classical Calderón-Zygmund theory, which yields the bound

$$\begin{aligned} \|D^{n-1} \Omega\|_{L^2(B_R)} &\leq C \|D_\Omega^{n-2} F_\Omega\|_{L^2(B_S)} \\ &\quad + C \sum_{\ell=0}^{n-2} \left(\frac{1}{(S-R)^{n-\ell-1}} \|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_S)} + \|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_S)}^{n/(\ell+1)} \right). \end{aligned} \quad (4.8)$$

In order to bound the terms of the last sum, we apply Gagliardo-Nirenberg interpolation in the form stated in Theorem 2.7, which gives

$$\|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_S)} \leq C \left(\|D^{n-1} \Omega\|_{L^2(B_S)}^{\ell/(n-1)} \|\Omega\|_{L^{2n}(B_S)}^{(n-\ell-1)/(n-1)} + \|\Omega\|_{L^{2n}(B_S)} \right)$$

for every $\ell \in \{0, \dots, n-2\}$. For a parameter $\varepsilon > 0$ to be chosen later, we twice apply Young's inequality to the right-hand side, which yields

$$\frac{1}{(S-R)^{n-\ell-1}} \|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_S)} \leq \varepsilon \|D^{n-1} \Omega\|_{L^2(B_S)} + \frac{C(\varepsilon)}{(S-R)^{n-1}} \|\Omega\|_{L^{2n}(B_S)},$$

as well as

$$\|D^\ell \Omega\|_{L^{2n/(\ell+1)}(B_S)}^{n/(\ell+1)} \leq \varepsilon \|D^{n-1} \Omega\|_{L^2(B_S)} + C(\varepsilon) \|\Omega\|_{L^{2n}(B_S)}^n.$$

We recall that $\|\Omega\|_{L^{2n}(B_S)} \leq C\kappa$ with $\kappa < 1$, so that we also may drop the exponent n in the last term. Plugging the two preceding estimates into (4.8), we obtain the bound

$$\begin{aligned} & \|D^{n-1} \Omega\|_{L^2(B_R)} \\ & \leq C\varepsilon \|D^{n-1} \Omega\|_{L^2(B_S)} + C \|D_\Omega^{n-2} F_\Omega\|_{L^2(B_1)} + \frac{C(\varepsilon)}{(S-R)^{n-1}} \|\Omega\|_{L^{2n}(B_1)}. \end{aligned}$$

Now we choose $\varepsilon > 0$ so small that $C\varepsilon \leq \frac{1}{2}$. Recalling that the above estimate holds for any $\frac{1}{2} \leq R < S \leq \frac{3}{4}$, we iterate it in a standard way (cf. [G, Lemma V.3.1]) to get

$$\|D^{n-1} \Omega\|_{L^2(B_{1/2})} \leq C \|D_\Omega^{n-2} F_\Omega\|_{L^2(B_1)} + C \|\Omega\|_{L^{2n}(B_1)}.$$

Combining this with (4.4), we arrive at

$$\|D^{n-1} \Omega\|_{L^2(B_{1/2})} + \|\Omega\|_{L^{2n}(B_1)} \leq C \|D_\Omega^{n-2} F_\Omega\|_{L^2(B_1)} + C \|F_\Omega\|_{L^n(B_1)}.$$

This implies the claimed estimate (4.3) by another application of the Gagliardo-Nirenberg interpolation estimate. \square

5 Removability of point singularities of the underlying bundle

For Yang-Mills connections, there exist removable singularity results in 4 dimensions [Uh1] and in higher dimensions [TT]. A related partial regularity result for Yang-Mills connections in higher dimensions has been established in [MR]. In our case, we do not work with Yang-Mills connections. Therefore it is important to note that Petrache and Rivière have removed the assumption of having a Yang-Mills connection from the removability result.

In this section, we prove a higher order version of their ground-breaking result [PR, Thm. 3.2] for higher dimensions. The crucial observation that enables Petrache and Rivière to construct a local trivialization without using the Yang-Mills equation is the continuity of gauge transformations between Coulomb gauges. This follows from estimates involving Lorentz spaces, a technique that has been introduced to Yang-Mills theory by Rivière in [Ri]. We generalize the arguments from [PR, Thm. 3.2], which are the $n = 2$ case of

Proposition 5.1 *Let $n \geq 2$, $U \subset \mathbb{R}^{2n}$ be a bounded domain. Assume that A and $B = u^{-1}Au + u^{-1}du$ are \mathfrak{g} -valued 1-forms of class $W^{n-1,2}$ satisfying*

$$d^*A = d^*B = 0 \quad \text{on } U.$$

(i) *Then the gauge change u is in $W_{\text{loc}}^{n,2} \cap C_{\text{loc}}^0(U, G)$, and for any $V \subset\subset U$, there is some constant $\bar{u} \in G$ depending on u , such that*

$$\begin{aligned} & \|u - \bar{u}\|_{W^{n,2} \cap C^0(V)} \\ & \leq C(\|A\|_{W^{n-1,2}(U)} + \|B\|_{W^{n-1,2}(U)} + \|A\|_{W^{n-1,2}(U)}^n + \|B\|_{W^{n-1,2}(U)}^n), \end{aligned}$$

where C depends only on n , U , V , and G .

(ii) *Moreover, there is a $\delta > 0$ depending on n , U , V , and G such that in the case*

$$\|A\|_{W^{n-1,2}(U)} + \|B\|_{W^{n-1,2}(U)} < \delta, \quad (5.1)$$

the following holds. For any $W \subset\subset V$, there is another gauge change $\tilde{u} \in W^{n,2} \cap C^0(U, G)$ that coincides with u on W and with the constant \bar{u} on $U \setminus V$, with the estimate

$$\begin{aligned} & \|\tilde{u} - \bar{u}\|_{W^{n,2} \cap C^0(U)} \\ & \leq C(\|A\|_{W^{n-1,2}(U)} + \|B\|_{W^{n-1,2}(U)} + \|A\|_{W^{n-1,2}(U)}^n + \|B\|_{W^{n-1,2}(U)}^n). \end{aligned}$$

The constant C here depends additionally on W .

Proof of (i), modelled after [PR, Prop. 3.5].

If no domain is indicated in integral norms, it is assumed to be U . By the Lorentz space version of the Sobolev embedding, we have $A, B \in L^{(2n,2)}$ and

$$\|A\|_{L^{(2n,2)}} \leq C\|A\|_{W^{n-1,2}}, \quad \|B\|_{L^{(2n,2)}} \leq C\|B\|_{W^{n-1,2}}.$$

Since

$$du = uB - Au,$$

and since $u \in L^\infty$ as it takes values in G , the previous estimates imply $du \in L^{(2n,2)}$ and

$$\|du\|_{L^{(2n,2)}} \leq C(\|A\|_{L^{(2n,2)}} + \|B\|_{L^{(2n,2)}}) \leq C(\|A\|_{W^{n-1,2}} + \|B\|_{W^{n-1,2}}).$$

Leibnitz' rule for functions h and 1-forms ω is $d^*(h\omega) = hd^*\omega - dh \cdot \omega$, and this carries over to the \mathfrak{g} -valued case. Using $d^*A = d^*B = 0$, we therefore find

$$-\Delta u = d^*(du) = d^*(uB - Au) = du \cdot A - B \cdot du.$$

On the right-hand side, we have products of $L^{(2n,2)}$ -functions, which are in $L^{(n,1)}$. Hence

$$\begin{aligned} \|\Delta u\|_{L^{(n,1)}} &\leq \|du\|_{L^{(2n,2)}} (\|A\|_{L^{(2n,2)}} + \|B\|_{L^{(2n,2)}}) \\ &\leq C(\|A\|_{W^{n-1,2}}^2 + \|B\|_{W^{n-1,2}}^2). \end{aligned} \quad (5.2)$$

Let η be a V - U -cutoff function. Then, writing u_0 for the mean value of u in some $\mathbb{R}^{k \times k} \supset G$,

$$\begin{aligned} \|\Delta(\eta(u - u_0))\|_{L^{(n,1)}} &\leq \|\Delta u\|_{L^{(n,1)}} + C(\|du\|_{L^{(n,1)}} + \|u - u_0\|_{L^{(n,1)}}) \\ &\leq \|\Delta u\|_{L^{(n,1)}} + C\|du\|_{L^{(2n,2)}} \\ &\leq C(\|A\|_{W^{n-1,2}} + \|B\|_{W^{n-1,2}} + \|A\|_{W^{n-1,2}}^2 + \|B\|_{W^{n-1,2}}^2) \\ &=: C(\dots), \end{aligned} \quad (5.3)$$

and $\eta(u - u_0) \equiv 0$ near ∂U . The standard elliptic estimate in Lorentz space gives

$$\|u - u_0\|_{W^{2,(n,1)}(V)} \leq C\|\eta(u - u_0)\|_{W^{2,(n,1)}(U)} \leq C(\dots).$$

Using $W^{2,(n,1)} \hookrightarrow L^\infty$ (which would not hold for $W^{2,n}$), we also have

$$\|u - u_0\|_{L^\infty(V)} \leq C(\dots).$$

In order to show that u is continuous, we use the estimate (5.2) with U replaced by an arbitrary ball $B_{2\rho}(x_0) \subset U$ and V replaced by $B_\rho(x_0)$. Abbreviating $u_\rho := \int_{B_{2\rho}(x_0)} u$ and bounding the lower order terms on the right-hand side by Hölder's inequality, we deduce

$$\begin{aligned} \|u - u_\rho\|_{L^\infty(B_\rho(x_0))} &\leq C \sum_{k=0}^{n-1} \left(\|D^k A\|_{L^{\frac{2n}{k+1}}(B_{2\rho}(x_0))} + \|D^k B\|_{L^{\frac{2n}{k+1}}(B_{2\rho}(x_0))} \right. \\ &\quad \left. + \|D^k A\|_{L^{\frac{2n}{k+1}}(B_{2\rho}(x_0))}^2 + \|D^k B\|_{L^{\frac{2n}{k+1}}(B_{2\rho}(x_0))}^2 \right). \end{aligned}$$

In this form, both sides of the inequality are scaling invariant, from which we infer that the constant C can be chosen independently from $\rho > 0$. Keeping in mind the Sobolev embedding $W^{n-1,2} \hookrightarrow W^{k,2n/(k+1)}$ that holds for any $k = 0, \dots, n-1$, we deduce $\|u - u_\rho\|_{L^\infty(B_\rho(x_0))} \rightarrow 0$ as $\rho \searrow 0$, from which we see that u is really continuous.

Observe that $u \in G$ everywhere, and hence

$$\text{dist}(u_0, G) \leq C(\dots).$$

This means that there is also $\bar{u} \in G$ such that

$$\|u - \bar{u}\|_{L^\infty(V)} \leq C(\dots).$$

Now that $u - \bar{u}$ is estimated in $W^{2,n} \cap L^\infty$, all that is missing are higher order estimates for u . Starting with $u \in W^{2,n} \cap L^\infty$, we plug that into $du = uB - Au$, and iterate that with any better result we achieve that way, consecutively finding (with V as domain of integration)

$$\begin{aligned} du &\in (W^{2,n} \cap L^\infty) \cdot W^{2,\frac{2n}{3}} \hookrightarrow W^{2,\frac{2n}{3}}, \\ du &\in (W^{3,\frac{2n}{3}} \cap L^\infty) \cdot W^{3,\frac{2n}{4}} \hookrightarrow W^{3,\frac{2n}{4}}, \\ &\vdots \\ du &\in (W^{n-1,\frac{2n}{n-1}} \cap L^\infty) \cdot W^{n-1,2} \hookrightarrow W^{n-1,2}. \end{aligned}$$

The last one implies $u \in W^{n,2}(V)$, with the asserted estimates, in which the powers of the norms build up because of iterated multiplication.

Proof of (ii). Now we assume that (5.1) holds for some sufficiently small $\delta > 0$. If we choose $\delta > 0$ small enough in dependence on n, U, V , and G , the estimate from (i) shows that the image of $\bar{u}^{-1}u|_V$ is contained in some neighborhood of $e \in G$ on which \exp_e^{-1} is defined and well behaved. More precisely, we can assume that the first n derivatives of \exp_e^{-1} and \exp_e on this neighborhood or its exponential image are bounded by a constant depending on G only. Writing $h(x) := \exp_e^{-1}(\bar{u}^{-1}u(x))$, and using a W - V -cutoff function $\eta : U \rightarrow [0, 1]$, we let

$$\tilde{u}(x) := \bar{u} \exp_e(\eta(x)h(x)),$$

and easily see that it has the asserted properties. \square

Remark. $A, B \in W^{1,2n}$ would not have been enough to infer $u \in C^0$, because $u \in W^{2,(n,1)} \hookrightarrow L^\infty$ is crucial, and we would only get $W^{2,(n,n/2)}$ instead. Therefore, we do need $A, B \in W^{n-1,2}$. For $n = 2$ (which is the case treated in [PR]), both conditions coincide.

Now we follow [PR] in proving the following removable singularity theorem.

Theorem 5.2 *Let P be a principal bundle over $B^{2n} \setminus \{0\}$. Assume we are given a connection $D_0 + A$ on P of class $W_{\text{loc}}^{n-1,2}(B^{2n} \setminus \{0\})$, for which*

$$\sum_{\ell=0}^{n-2} \|D_A^\ell F_A\|_{L^{2n/(\ell+2)}} < \infty.$$

Then there exists a gauge of class $W_{\text{loc}}^{n,2}$ in which the bundle extends to a smooth bundle over B^{2n} , and the connection extends to a connection in $W^{n-1,2}(B^{2n})$ in this gauge.

For what follows, we define four sequences of spherical shells,

$$\begin{aligned} Q_k &:= B_{2^{-8k-3}} \setminus B_{2^{-8k-14}}, \\ R_k &:= B_{2^{-8k-2}} \setminus B_{2^{-8k-15}}, \\ S_k &:= B_{2^{-8k-1}} \setminus B_{2^{-8k-16}}, \\ T_k &:= B_{2^{-8k}} \setminus B_{2^{-8k-17}} \end{aligned}$$

for all $k \in \mathbb{N} \cup \{0\}$. Note $Q_k \subset R_k \subset S_k \subset T_k$.

Lemma 5.3 *There exists a constant $\delta \in (0, 1)$ such that*

$$\|D_A^{n-2}F_A\|_{L^2(T_k)} + \|F_A\|_{L^n(T_k)} < \delta$$

*implies that the bundle E is trivial over Q_k , and that in a suitable gauge the connection is represented by a $W^{n-1,2}$ -form A_k with $d^*A_k = 0$ and the estimate*

$$\sum_{\ell=0}^{n-1} \|D^\ell A_k\|_{L^{\frac{2n}{\ell+1}}(Q_k)} \leq C(\|D_A^{n-2}F_A\|_{L^2(T_k)} + \|F_A\|_{L^n(T_k)}). \quad (5.4)$$

Here, the constants δ and C depend only on M and G , and in particular not on k .

Proof, following [PR, Lemma 3.6].

By scaling invariance, it is enough to prove the lemma for $k = 0$. We cover S_0 by two charts S_+ and S_- both diffeomorphic to B^{2n} , e.g.

$$S_+ := \{x \in S_0 : x_{2n} > -2^{-18}\}, \quad S_- := \{x \in S_0 : x_{2n} < 2^{-18}\}.$$

On both S_+ and S_- , we can apply the higher order Uhlenbeck estimates from Theorem 4.1, which clearly hold also on domains diffeomorphic to a ball, with a constant additionally depending on the diffeomorphism. Because of the scaling invariance of the assertion (5.4), we have to choose only two diffeomorphisms for S_+ and S_- before applying Theorem 4.1, which means that the additional constant will depend only on n . Hence we can assume that the connection is represented by A_+ on S_+ and by A_- on S_- such that

$$\begin{aligned} &\sum_{j=0}^{n-1} \left(\|D^j A_+\|_{L^{\frac{2n}{j+1}}(S_+)} + \|D^j A_-\|_{L^{\frac{2n}{j+1}}(S_-)} \right) \\ &\leq C(\|D_A^{n-2}F_A\|_{L^2(T_0)} + \|F_A\|_{L^n(T_0)}) \leq C\delta. \end{aligned}$$

Now we let $R_+ := R_0 \cap S_+$, $R_- := R_0 \cap S_-$, and

$$R_{++} := \{x \in R_0 : x_{2n} > -2^{-19}\}, \quad R_{--} := \{x \in R_0 : x_{2n} < 2^{-19}\}.$$

Note that in particular $R_{++} \subset R_+$ and $R_{--} \subset R_-$. Proposition 5.1 gives us $A_+ = u^{-1}du + u^{-1}A_-u$ for some gauge transformation u controlled in $W_{\text{loc}}^{n,2} \cap$

$C_{\text{loc}}^0(S_+ \cap S_-)$. Part (ii) provides us with a modification $\tilde{u} \in W^{n,2} \cap C^0(R_{--}, G)$ coinciding with u on $R_{++} \cap R_{--}$ and with some constant $\bar{u} \in G$ on $R_{--} \setminus R_+$, such that

$$\|\tilde{u} - \bar{u}\|_{W^{n,2} \cap C^0(R_{--})} \leq C(\|D_A^{n-2} F_A\|_{L^2(T_0)} + \|F_A\|_{L^n(T_0)}) \leq C\delta. \quad (5.5)$$

Now here is a representative of the connection on all of R_0 :

$$\tilde{A}_0(x) := \begin{cases} A_+(x) & \text{if } x_{2n} \geq 0, \\ \tilde{u}^* A_-(x) & \text{if } x_{2n} < 0. \end{cases} \quad (5.6)$$

It is important to say how to interpret that assertion. Originally, the connection described by A_+ and A_- is given on a bundle over R_0 which is glued together along $R_0 \cap \{x_{2n} = 0\}$ with the transition function $\tilde{u} = u$. By the estimate (5.5), u takes its values in a small ball around \bar{u} in G and therefore represents the trivial homotopy class in $[R_0 \cap \{x_{2n} = 0\}, G] \cong \pi_{2n-2}(G)$. Since R_0 retracts to S^{2n-1} , the G -principal fibre bundles over R_0 are classified as those over S^{2n-1} , that is by the element of $\pi_{2n-2}(G)$ that the transition map, when restricted to an equator, represents. See [Na, Section 4.4] for details on the classification. Now u represents the trivial class, hence the original bundle must have been trivial, and (5.6) expresses the connection in a trivialization of the bundle, where the transition map is the identity. The bounds from (5.5) of \tilde{u} , together with the estimates for A_+ and A_- , show

$$\sum_{j=0}^{n-1} \|D^j \tilde{A}_0\|_{L^{\frac{2n}{j+1}}(R_0)} \leq C(\|D_A^{n-2} F_A\|_{L^2(T_0)} + \|F_A\|_{L^n(T_0)}) \leq C\delta. \quad (5.7)$$

That is almost all we would require from A_0 , but \tilde{A}_0 is not yet in Coulomb gauge, $d^* \tilde{A}_0$ may be $\neq 0$. Trying to gauge the connection again, we cannot apply Uhlenbeck's gauge theorem directly, since we do not know if it still holds on domains which are not diffeomorphic to B^{2n} . We therefore modify \tilde{A}_0 to find a connection on a ball, namely \hat{A}_0 on $B_{1/4}$ given by

$$\hat{A}_0 := \eta \tilde{A}_0,$$

where here η is a radial cutoff function which is $\equiv 1$ on Q_0 and $\equiv 0$ on $B_{1/4} \setminus R_0$ (where \tilde{A} is undefined). Then, by (5.7), we have

$$\|F_{\hat{A}_0}\|_{L^n(B_{1/4})} \leq C(\|F_A\|_{L^n(R_0)} + \|\tilde{A}_0\|_{L^n(R_0)} + \|\tilde{A}_0\|_{L^{2n}(R_0)}^2) \leq C\delta,$$

and if δ has been chosen small enough, we may apply Uhlenbeck's gauge theorem to find a gauge-transformed version A_0 of \hat{A}_0 with $d^* A_0 = 0$. And since $\hat{A}_0 = \tilde{A}_0$ on Q_0 , A_0 represents our originally given connection on Q_0 . The asserted estimate for A_0 follows again from Theorem 4.1. \square

The **proof of Theorem 5.2** again follows the arguments outlined in [PR]. By restricting to a very small ball B_r centered at 0, we may assume that the integrals of $|F_A|^n$ and $|D_A^{n-2}F_A|^2$ are as small as we want. Both of them are scaling invariant, hence we may rescale to B^{2n} and assume

$$\|D_A^{n-2}F_A\|_{L^2(B^{2n})} + \|F_A\|_{L^n(B^{2n})} < \delta,$$

for a small $\delta \in (0, 1)$ yet to be chosen. Letting

$$\delta_k := \|D_A^{n-2}F_A\|_{L^2(T_k)} + \|F_A\|_{L^n(T_k)},$$

we have

$$\begin{aligned} \sum_{k=0}^{\infty} \delta_k^n &\leq C \sum_{k=0}^{\infty} \|D_A^{n-2}F_A\|_{L^2(T_k)}^2 + C \sum_{k=0}^{\infty} \|F_A\|_{L^n(T_k)}^n \\ &\leq C(\|D_A^{n-2}F_A\|_{L^2(B^{2n})}^2 + \|F_A\|_{L^n(B^{2n})}^n) \\ &\leq C\delta^2. \end{aligned} \tag{5.8}$$

If δ is small enough, we can apply Lemma 5.3 to find forms A_k representing our connection on Q_k and satisfying $d^*A_k = 0$ as well as the estimates

$$\sum_{j=0}^{n-1} \|D^j A_k\|_{L^{\frac{2n}{j+1}}(Q_k)} \leq C\delta_k. \tag{5.9}$$

Let $W_k := B_{2^{-8k-12}} \setminus B_{2^{-8k-13}}$ be a shell strictly inside $U_k := Q_k \cap Q_{k+1} = B_{2^{-8k-11}} \setminus B_{2^{-8k-14}}$. Using Proposition 5.1 (ii) (which (5.9) allows us), we find mappings $\tilde{u}_k \in W^{n,2} \cap C^0(U_k, G)$ and constants $\bar{u}_k \in G$ such that

$$\begin{aligned} A_{k+1} &= \tilde{u}_k^* A_k \quad \text{on } W_k, \\ \tilde{u}_k &\equiv \bar{u}_k \quad \text{near } \partial U_k, \\ \|\tilde{u}_k - \bar{u}_k\|_{W^{n,2} \cap C^0(U_k)} &\leq C\delta_k. \end{aligned}$$

Defining

$$\bar{v}_k := \bar{u}_{k-1}^{-1} \dots \bar{u}_1^{-1} \bar{u}_0^{-1}, \quad \Omega_k := \bar{v}_k^* A_k,$$

we have

$$\Omega_{k+1} = u_k^* \Omega_k \text{ on } W_k, \text{ where } u_k := \bar{v}_k^{-1} \tilde{u}_k \bar{u}_k^{-1} \bar{v}_k. \tag{5.10}$$

The modified gauge changes u_k satisfy

$$u_k - e = \bar{v}_k^{-1} (\tilde{u}_k - \bar{u}_k) \bar{u}_k^{-1} \bar{v}_k,$$

and hence

$$\|u_k - e\|_{W^{n,2} \cap C^0(U_k)} = \|\tilde{u}_k - \bar{u}_k\|_{W^{n,2} \cap C^0(U_k)} \leq C\delta_k \rightarrow 0. \tag{5.11}$$

Together with (5.9), this implies

$$\sum_{j=0}^{n-1} \|D^j \Omega_k\|_{L^{\frac{2n}{j+1}}(U_k)} \leq C\delta_k. \quad (5.12)$$

Not only are the u_k close to the identity of G , but even $u_k \equiv e$ near ∂U_k . Together with (5.10), this means that

$$\Omega := \begin{cases} \Omega_k & \text{on every } B_{2^{-8k-6}} \setminus B_{2^{-8k-11}}, \\ u_k^* \Omega_k & \text{on every } B_{2^{-8k-11}} \setminus B_{2^{-8k-14}} \end{cases}$$

defines a connection form Ω on $B_{1/32} \setminus \{0\}$ which is locally (that is, away from 0) in $W^{n-1,2}$. This is because u_k has been constructed carefully in order to prevent Ω from having jumps across spheres. And Ω represents the originally given connection in some gauge, since it is obtained from the A_k by a sequence of gauge transformations. Moreover, it is in $W^{1,n}$ even across 0, because summing up the contributions from (5.11) and (5.12) for each k gives

$$\|\Omega\|_{W^{1,n}(B_{1/64})}^n \leq \sum_{k=0}^{\infty} \|\Omega\|_{W^{1,n}(Q_k)}^n \leq C \sum_{k=0}^{\infty} \delta_k^n \leq C\delta^2,$$

where here, the last estimate follows from (5.8). This means that we can interpret Ω as a $W^{1,n}$ -connection form on the trivial bundle over $B_{1/64}$, hence we have removed the singularity of the bundle. A final application of the Uhlenbeck gauge theorem 4.1 provides us with another gauge that transforms Ω into the desired $W^{n-1,2}$ -connection on the trivial bundle over $B_{1/128}$. \square

6 Euler-Lagrange equations

The bi-Yang-Mills equation has been computed by [BU], to the effect that the Euler-Lagrange equation of $\int_M |d_A^* F_A|^2 dx$ is of the form

$$d_A^* d_A d_A^* F_A = d_A^* F_A \# F_A,$$

where here “ $\#$ ” is used as before. We will generalize this here, first for $\int_M |d_A^{*\wedge n-2} F_A|^2 dx$ and then for Y_n , and for Z_n by analogy. As a preparation, we compute

Lemma 6.1 *Let M be a manifold of dimension m , and $n \geq 2$ be given. Let A_t be a smooth 1-parameter family of $W^{n-1,2}$ -connections on a principal bundle P over M and $A := A_0$, $\alpha := \frac{d}{dt}|_{t=0} A_t$. Then we have*

$$\begin{aligned} \frac{d}{dt}|_{t=0} d_{A_t}^{*\wedge n-2} F_{A_t} &= d_A^{\wedge n-1} \alpha - (-1)^m \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} d_A^{\wedge n-3-2k} * [\alpha, * d_A^{*\wedge 2k} F_A] \\ &\quad + \sum_{k=0}^{\lfloor \frac{n-4}{2} \rfloor} d_A^{*\wedge n-4-2k} [\alpha, d_A^{*\wedge 2k+1} F_A]. \end{aligned} \quad (6.1)$$

Proof. We proceed by induction over n . For $n = 2$, we compute

$$\frac{d}{dt}|_{t=0} F_{A_t} = \frac{d}{dt}|_{t=0} (dA_t + \frac{1}{2}[A_t, A_t]) = d\alpha + [A, \alpha] = d_A \alpha.$$

For the inductive step, we observe that for $n \geq 2$ even,

$$\begin{aligned} \frac{d}{dt}|_{t=0} d_{A_t}^{*\wedge n-1} F_{A_t} &= \frac{d}{dt}|_{t=0} (d_{A_t}^* d_{A_t}^{*\wedge n-2} F_{A_t}) \\ &= d_A^* \left(\frac{d}{dt}|_{t=0} d_{A_t}^{*\wedge n-2} F_{A_t} \right) + \left(\frac{d}{dt}|_{t=0} d_{A_t}^* \right) d_A^{*\wedge n-2} F_A \\ &= d_A^* \left(\frac{d}{dt}|_{t=0} d_{A_t}^{*\wedge n-2} F_{A_t} \right) - (-1)^m * [\alpha, *d_A^{*\wedge n-2} F_A], \end{aligned}$$

while for odd $n \geq 3$, similarly,

$$\frac{d}{dt}|_{t=0} d_{A_t}^{*\wedge n-1} F_{A_t} = d_A \left(\frac{d}{dt}|_{t=0} d_{A_t}^{*\wedge n-2} F_{A_t} \right) + [\alpha, d_A^{*\wedge n-2} F_A].$$

The assertion now follows easily. □

Lemma 6.2 (Euler-Lagrange equations) *Let M be of any dimension m , and $n \geq 2$ be given. The Euler-Lagrange equation for $\int_M |d_A^{*\wedge n-2} F_A|^2 dx$ is given by*

$$d_A^{*\wedge 2n-3} F_A = \sum_{\ell=0}^{n-3} d_A^{*\wedge 2n-5-\ell} F_A \# d_A^{*\wedge \ell} F_A,$$

with bilinear forms noted as above. The Euler-Lagrange equation for Y_n reads

$$d_A^{*\wedge 2n-3} F_A + \frac{n}{2} d_A^* (|F_A|^{n-2} F_A) = \sum_{\ell=0}^{n-3} d_A^{*\wedge 2n-5-\ell} F_A \# d_A^{*\wedge \ell} F_A.$$

Proof. Assume that A is a critical point of $\int_M |d_A^{*\wedge n-2} F_A|^2 dx$ and A_t a variation of A as in the Lemma 6.1. The lemma then yields

$$\begin{aligned}
0 &= \frac{1}{2} \frac{d}{dt} \Big|_{t=0} \int_M |d_{A_t}^{*\wedge n-2} F_{A_t}|^2 dx \\
&= \int_M \langle d_A^{*\wedge n-2} F_A, d_A^{\wedge n-1} \alpha \rangle dx \\
&\quad - (-1)^m \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} \int_M \langle d_A^{*\wedge n-2} F_A, d_A^{\wedge n-3-2k} * [\alpha, *d_A^{*\wedge 2k} F_A] \rangle dx \\
&\quad + \sum_{k=0}^{\lfloor \frac{n-4}{2} \rfloor} \int_M \langle d_A^{*\wedge n-2} F_A, d_A^{*\wedge n-4-2k} [\alpha, d_A^{*\wedge 2k+1} F_A] \rangle dx \\
&= \int_M \langle d_A^{*\wedge 2n-3} F_A, \alpha \rangle dx \\
&\quad + \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} \int_M \langle *d_A^{*\wedge 2n-5-2k} F_A, [\alpha, *d_A^{*\wedge 2k} F_A] \rangle dx \\
&\quad + \sum_{k=0}^{\lfloor \frac{n-4}{2} \rfloor} \int_M \langle d_A^{*\wedge 2n-6-2k} F_A, [\alpha, d_A^{*\wedge 2k+1} F_A] \rangle dx,
\end{aligned}$$

from which we read off the assertion. The additional term for Y_n is straightforward. \square

While Lemma 6.2 writes the Euler-Lagrange equation in a rather simple form, what we need is some divergence form to make sense of weak solutions.

Lemma 6.3 (Euler-Lagrange equations, divergence form) *Let M be of any dimension m , and $n \geq 2$ be given. Then we have smooth coefficient forms P_k ($k \in \{n+1, \dots, 2n-1\}$) such that $P_k[A]$ depends on $A, DA, \dots, D^{n-1}A$ and is a 1-form if k is odd, a 2-form if k is even, and that the Euler-Lagrange equation for $\int_M |d_A^{*\wedge n-2} F_A|^2 dx$ reads*

$$d^{\wedge(2n-2)} A = \sum_{k=n+1}^{2n-1} d^{(*)\wedge(2n-1-k)} P_k[A]. \quad (6.2)$$

If A is in Coulomb gauge ($d^* A = 0$), this can be re-written as

$$(-1)^{n-1} \Delta^{n-1} A = \sum_{k=n+1}^{2n-1} d^{(*)\wedge(2n-1-k)} P_k[A]. \quad (6.3)$$

Here we wrote $d^{(*)\wedge i}$ for $d^{\wedge i}$ if operating on 1-forms, respectively for $d^{*\wedge i}$ if acting

on 2-forms, cf. Section 2.2 for the definition. Each $P_k[A]$ satisfies

$$|P_k[A]| \leq C \left(1 + \sum_{j=0}^{n-1} |D^j A|^{\frac{k}{j+1}} \right) \quad \text{for } k \in \{n+1, \dots, 2n-1\}. \quad (6.4)$$

For the functional Y_n , the same formulae (6.2)–(6.4) hold, with different forms P_{2n-2} and P_{2n-1} of the same structure.

For the functional Z_n , (6.2) and (6.3) have to be modified and read

$$\Delta^{n-2} d^* dA = \sum_{k=n+1}^{2n-1} (D^*)^{2n-1-k} P_k[A] \quad (6.5)$$

for every gauge, and

$$-\Delta^{n-1} A = \sum_{k=n+1}^{2n-1} (D^*)^{2n-1-k} P_k[A] \quad (6.6)$$

in Coulomb gauge. In this case, the $P_k[A]$ are sections of $\otimes^k T^*M \otimes \mathfrak{g}_P$, and again, the estimates (6.4) hold.

Proof. Again we apply Lemma 6.1 to critical points of $\int_M |d_A^{*\wedge n-2} F_A|^2 dx$, but this time after differentiating through all the terms of the lemma.

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d}{dt} \Big|_{t=0} \int_M |d_{A_t}^{*\wedge n-2} F_{A_t}|^2 dx \\ &= \int_M \left\langle d_A^{*\wedge n-2} F_A, d_A^{\wedge n-1} \alpha \right\rangle dx \\ &\quad + \int_M \left\langle d_A^{*\wedge n-2} F_A, \sum_{j \in K_n} d_A^{\wedge j_1-1} \alpha \# (d_A^{*\wedge j_2-2} F_A \# \dots \# d_A^{*\wedge j_\ell-2} F_A) \right\rangle dx. \end{aligned} \quad (6.7)$$

Here

$$K_n := \{j = (j_1, \dots, j_\ell) : \ell \geq 2, j_1 \geq 1, j_2 \geq \dots \geq j_\ell \geq 2, j_1 + \dots + j_\ell \leq n\}.$$

The only reason that K_n contains sets with $j_1 + \dots + j_\ell < n$ is that such terms appear when the forms “ $\#$ ” themselves are differentiated.

We note that by the definition of K_n , we have $j_1 - 1 \in \{0, \dots, n-3\}$, note, however, that in the first term on the right-hand side of (6.7) there is an $(n-1)$ -th derivative of α . Therefore, the above equation can be transformed into

$$\int_M \langle d^{\wedge n-1} A, d^{\wedge n-1} \alpha \rangle + \sum_{k=n+1}^{2n-1} \langle d^{\wedge (2n-1-k)} \alpha, P_k[A] \rangle dx = 0 \quad (6.8)$$

for functions $P_k[A]$ of the form

$$P_k[A] = \sum_{j_1 + \dots + j_\ell \leq k} D^{j_1-1} A \# \dots \# D^{j_\ell-1} A.$$

The claimed estimate (6.4) follows by applying Young's inequality with exponents $\frac{k}{j_1}, \dots, \frac{k}{j_\ell}$ (and then again Young if the exponents are too small). The equation (6.8) is the weak formulation of the assertion (6.2). If A is in Coulomb gauge, then $(-1)^{n-1} \Delta^{n-1} A = d^{\wedge(2n-2)} A$, which yields (6.3).

For Y_n , the additional term $\frac{n}{2} d_A^* (|F_A|^{n-2} F_A)$ contributes to both P_{2n-2} and P_{2n-1} provided $n > 2$, while for $n = 2$, it coincides with the term $d_A^* \wedge^{2n-3} F_A$ that was already treated above.

Finally, for Z_n , everything works quite similarly, replacing alternating powers of d_A and d_A^* by suitable combinations of D_A and D_A^* . Note that the leading term in Lemma 6.2 becomes $d_A^* (D_A^*)^{n-2} D_A^{n-2} F_A$. \square

7 Smoothness of weak solutions

Next we apply elliptic bootstrap arguments to establish smoothness of solutions for the Euler-Lagrange equations of Y_n or Z_n , or many similar functionals. Note that in particular our result includes regularity of weakly bi-Yang-Mills connections in the sense of [BU], in dimensions $m \leq 6$.

Theorem 7.1 *Assume that $A \in W^{n-1,2}(B_1^m, \wedge^1 \mathbb{R}^m \otimes \mathfrak{g})$ is a weak solution of (6.2) or (6.5), and that it is in Coulomb gauge, i. e. $d^* A = 0$. We suppose that (6.4) is in force and that $m \leq 2n$. Then there holds $A \in C^\infty(B_1, \wedge^1 \mathbb{R}^m \otimes \mathfrak{g})$.*

Proof. The proof consists of three steps. We begin with

Step 1: A Morrey-type estimate. The goal in this step is to establish a Morrey-type estimate of the type

$$\sum_{j=0}^{n-1} \sup_{B_\rho(y) \subset B_R} \rho^{-2\alpha} \left(\rho^{2n-m} \int_{B_\rho} |D^j A|^{\frac{2n}{j+1}} dx \right)^{\frac{j+1}{n}} < \infty \quad (7.1)$$

for any $\alpha \in (0, 1)$ and every $R > 0$. We note that in the subcritical case $m < 2n$, this estimate is trivially satisfied at least for some $\alpha \in (0, 1)$, but this first step of the proof is crucial in the critical case $m = 2n$.

We temporarily fix $y \in B_R$. For an $\varepsilon_0 \in (0, 1)$ to be fixed later, we choose $r_0 \in (0, 1)$ small enough to achieve $B_{r_0}(y) \subset B_R$ and

$$\Phi(r_0) := \sum_{j=0}^{n-1} \left(r_0^{2n-m} \int_{B_{r_0}(y)} |D^j A|^{\frac{2n}{j+1}} dx \right)^{\frac{j+1}{n}} < \varepsilon_0. \quad (7.2)$$

For the remainder of this first step, we always consider a radius $r \in (0, r_0)$ and abbreviate $B_r := B_r(y) \subset B_R$. We decompose the solution into $A = A_0 + A_1$, where A_1 is the Δ^{n-1} -polyharmonic function (component-wise) with the same

boundary values as A . Since $A_0 \in W_0^{n-1,2}(B_r)$, A solves the Euler-Lagrange equation and $\Delta^{n-1}(A - A_0) = 0$, there holds

$$\begin{aligned}
& r^{2n-m} \int_{B_r} |D^{n-1} A_0|^2 dx \\
& \leq C r^{2n-m} \sum_{k+1}^{2n-1} \int_{B_r} |P_k[A]| |D^{2n-1-k} A_0| dx \\
& \leq C \sum_{k+1}^{2n-1} \left(r^{2n-m} \int_{B_r} |P_k[A]|^{\frac{2n}{k}} dx \right)^{\frac{k}{2n}} \left(r^{2n-m} \int_{B_r} |D^{2n-1-k} A_0|^{\frac{2n}{2n-k}} dx \right)^{\frac{2n-k}{2n}} \\
& \leq C \sum_{k+1}^{2n-1} \left(r^{2n-m} \int_{B_r} |P_k[A]|^{\frac{2n}{k}} dx \right)^{\frac{k}{2n}} \left(r^{2n-m} \int_{B_r} |D^{n-1} A_0|^2 dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Here, we used the Sobolev embedding $W_0^{n-1,2} \hookrightarrow W^{2n-1-k, \frac{2n}{2n-k}}$ for the last step, which holds in any dimension $m \leq 2n$. Re-absorbing the last integral on the left-hand side and using the Poincaré-Sobolev inequality, we arrive at

$$\begin{aligned}
\sum_{j=0}^{n-1} \left(r^{2n-m} \int_{B_r} |D^j A_0|^{\frac{2n}{j+1}} dx \right)^{\frac{j+1}{n}} & \leq C r^{2n-m} \int_{B_r} |D^{n-1} A_0|^2 dx \\
& \leq C \sum_{k+1}^{2n-1} \left(r^{2n-m} \int_{B_r} |P_k[A]|^{\frac{2n}{k}} dx \right)^{\frac{k}{n}}.
\end{aligned} \tag{7.3}$$

Since $D^j A_1 \in C^\infty(B_r)$ is polyharmonic for every $j \in \mathbb{N}$, we get similarly as in [GS, Lemma 6.2]

$$\begin{aligned}
\int_{B_\rho} |D^j A_1|^{\frac{2n}{j+1}} dx & \leq C \|D^j A_1\|_{L^\infty(B_{r/2})}^{\frac{2n}{j+1}} \\
& \leq C \left(\int_{B_r} |D^j A_1| dx \right)^{\frac{2n}{j+1}} \leq C \int_{B_r} |D^j A_1|^{\frac{2n}{j+1}} dx
\end{aligned} \tag{7.4}$$

for all $\rho \in (0, \frac{r}{2})$. Now we first apply (7.4) with $j \in \{0, \dots, n-1\}$ and then (7.3), with the result

$$\begin{aligned}
& \rho^{2n-m} \int_{B_\rho} |D^j A|^{\frac{2n}{j+1}} dx \\
& \leq C \rho^{2n-m} \int_{B_\rho} |D^j A_0|^{\frac{2n}{j+1}} dx + C \rho^{2n} \int_{B_\rho} |D^j A_1|^{\frac{2n}{j+1}} dx \\
& \leq C \rho^{2n-m} \int_{B_\rho} |D^j A_0|^{\frac{2n}{j+1}} dx + C \rho^{2n} \int_{B_r} |D^j A_1|^{\frac{2n}{j+1}} dx \\
& \leq C r^{2n-m} \int_{B_r} |D^j A_0|^{\frac{2n}{j+1}} dx + C \left(\frac{\rho}{r}\right)^{2n} r^{2n-m} \int_{B_r} |D^j A|^{\frac{2n}{j+1}} dx \\
& \leq C \sum_{k+1}^{2n-1} \left(r^{2n-m} \int_{B_r} |P_k[A]|^{\frac{2n}{k}} dx \right)^{\frac{k}{j+1}} + C \left(\frac{\rho}{r}\right)^{2n} r^{2n-m} \int_{B_r} |D^j A|^{\frac{2n}{j+1}} dx
\end{aligned}$$

and after taking roots, we infer

$$\begin{aligned}
& \left(\rho^{2n-m} \int_{B_\rho} |D^j A|^{\frac{2n}{j+1}} dx \right)^{\frac{j+1}{n}} \\
& \leq C \sum_{k+1}^{2n-1} \left(r^{2n-m} \int_{B_r} |P_k[A]|^{\frac{2n}{k}} dx \right)^{\frac{k}{n}} \\
& \quad + C \left(\frac{\rho}{r}\right)^{2(j+1)} \left(r^{2n-m} \int_{B_r} |D^j A|^{\frac{2n}{j+1}} dx \right)^{\frac{j+1}{n}}
\end{aligned} \tag{7.5}$$

for every $j \in \{0, \dots, n-1\}$. By the bounds (6.4) for the functions P_k , we can estimate

$$\begin{aligned}
\left(r^{2n-m} \int_{B_r} |P_k[A]|^{\frac{2n}{k}} dx \right)^{\frac{k}{n}} & \leq C \left(\sum_{j=0}^{n-1} r^{2n-m} \int_{B_r} |D^j A|^{\frac{2n}{j+1}} dx + r^{2n} \right)^{\frac{k}{n}} \\
& \leq C \sum_{j=0}^{n-1} \Phi(r)^{\frac{k}{j+1}} + C r^{2k} \leq C \varepsilon_0^{1/n} \Phi(r) + C r^{2(n+1)}.
\end{aligned}$$

In the last step, we used the fact that $k \geq n+1$ and $j+1 \leq n$ in each term of the preceding sum and the property (7.2). Combining the two preceding estimates for $\rho = \theta r$ and summing over $j = \{0, \dots, n-1\}$, we arrive at

$$\Phi(\theta r) \leq C(\varepsilon_0^{1/n} + \theta^2) \Phi(r) + C r^{2(n+1)}$$

for every $\theta \in (0, \frac{1}{2})$. For an arbitrary $\alpha \in (0, 1)$, we now choose $\theta \in (0, \frac{1}{2})$ such that $C\theta^2 \leq \frac{1}{2}\theta^{2\alpha}$ and then $\varepsilon_0 \in (0, 1)$ so small that $C\varepsilon_0^{1/n} \leq \frac{1}{2}\theta^{2\alpha}$. With these choices of parameters, the preceding bound reads $\Phi(\theta r) \leq \theta^{2\alpha} \Phi(r) + C r^{2(n+1)}$, which can be iterated to give

$$\Phi(\rho) \leq C(\alpha) \left[\left(\frac{\rho}{r_0}\right)^{2\alpha} \Phi(r_0) + \rho^{2\alpha} \right] \tag{7.6}$$

for every $\rho \in (0, r_0)$, where $\alpha \in (0, 1)$ can be chosen arbitrarily. This proves the assertion (7.1).

Step 2: $C^{k, \alpha}$ -regularity for all $k < n - 1$. In this step, we wish to improve the Morrey estimate from the preceding step to

$$\sum_{j=[\alpha]}^{n-1} \sup_{B_\rho(y) \subset B_R} \rho^{-2\alpha} \left(\rho^{2n-m} \int_{B_\rho(y)} |D^j A|^{\frac{2n}{j+1}} dx \right)^{\frac{j+1}{n}} < \infty \quad (7.7)$$

for every $\alpha \in (0, n)$ and every $R < 1$. Here, $[\alpha]$ denotes the largest integer smaller than or equal to α . If (7.7) did not hold, we would have

$$\bar{\alpha} := \sup\{\alpha \in (0, n) : (7.7) \text{ is valid for } \alpha\} < n.$$

We note that according to the Morrey-type estimate (7.6) established in Step 1, the claim holds true for every $\alpha \in (0, 1)$, and consequently, $\bar{\alpha} \geq 1$. In order to derive a contradiction, we choose an exponent $\alpha \notin \mathbb{N}$ with $0 < \alpha < \bar{\alpha} < \alpha(1 + \frac{1}{n}) < n$. Since $\alpha < \bar{\alpha}$, we know that (7.7) is valid for this value of α , which implies in particular that

$$\int_{B_\rho(y)} |D^{[\alpha]} A|^{\frac{2n}{[\alpha]+1}} dx \leq \widehat{C} \rho^{\alpha \frac{2n}{[\alpha]+1} - 2n} \quad \text{for all } B_\rho(y) \subset B_R$$

for every $R < 1$. Here and in the rest of the proof, we follow the convention to write C for constants that depend at most on n and \mathfrak{g} and \widehat{C} for constants that may additionally depend on R and A . Since α was chosen as non-integer, the exponent on the right-hand side of the preceding estimate satisfies $\alpha \frac{2n}{[\alpha]+1} - 2n > [\alpha] \frac{2n}{[\alpha]+1} - 2n = -\frac{2n}{[\alpha]+1}$. Therefore, the Dirichlet growth theorem implies $A \in C_{\text{loc}}^{j, \gamma}(B_1)$ for every $j \in \{0, \dots, [\alpha] - 1\}$ and some $\gamma > 0$, which implies in particular

$$\sum_{j=0}^{[\alpha]-1} \int_{B_r} |D^j A|^{\frac{2n}{j+1}} dx \leq \widehat{C} r^m, \quad (7.8)$$

for every ball $B_r \subset B_R$, where the constant \widehat{C} might depend on the $C^{[\alpha]-1}$ -norm of A on B_R . Now we use the estimates (6.4) for the functions P_k , together with (7.7) and (7.8) in order to deduce that there holds for every $k \in \{n+1, \dots, 2n-1\}$

$$\begin{aligned} \left(r^{2n-m} \int_{B_r} |P_k[A]|^{\frac{2n}{k}} dx \right)^{\frac{k}{n}} &\leq C \left(\sum_{j=0}^{n-1} r^{2n-m} \int_{B_r} |D^j A|^{\frac{2n}{j+1}} dx + r^{2n} \right)^{\frac{k}{n}} \\ &\leq \widehat{C} r^{2k} + \widehat{C} \sum_{j=[\alpha]}^{n-1} r^{2\alpha \frac{k}{j+1}} \\ &\leq \widehat{C} r^{2(n+1)} + \widehat{C} r^{2\alpha(1+\frac{1}{n})} \leq \widehat{C} r^{2\alpha(1+\frac{1}{n})} \end{aligned} \quad (7.9)$$

since $r < 1$ and $\alpha(1 + \frac{1}{n}) < n$ by the choice of α . Combining this with the excess decay estimate (7.5), we infer for every $j \in \{0, \dots, n-1\}$

$$\begin{aligned} & \left(\rho^{2n-m} \int_{B_\rho} |D^j A|^{\frac{2n}{j+1}} dx \right)^{\frac{j+1}{n}} \\ & \leq C \left(\frac{\rho}{r} \right)^{2(j+1)} \left(r^{2n-m} \int_{B_r} |D^j A|^{\frac{2n}{j+1}} dx \right)^{\frac{j+1}{n}} + \widehat{C} r^{2\alpha(1+\frac{1}{n})} \end{aligned} \quad (7.10)$$

for every $\rho \in (0, \frac{r}{2})$. As long as $2\alpha(1 + \frac{1}{n}) < 2(j+1)$, i. e. for $j > \alpha(1 + \frac{1}{n}) - 1$, we can iterate this to get

$$\left(\rho^{2n-m} \int_{B_\rho(y)} |D^j A|^{\frac{2n}{j+1}} dx \right)^{\frac{j+1}{n}} \leq \widehat{C} \rho^{2\alpha(1+\frac{1}{n})},$$

for every $B_\rho(y) \subset B_R$, where \widehat{C} as before might depend on A and on R , but not on ρ . But this implies (7.7) for $\alpha(1 + \frac{1}{n}) > \bar{\alpha}$ instead of α , which is a contradiction to the choice of $\bar{\alpha}$. We conclude that the claim (7.7) holds true for every $\alpha \in (0, n)$. This implies in particular $A \in C_{\text{loc}}^{n-2, \alpha}(B_1)$ by the Dirichlet growth theorem.

Step 3: $C^{n-1, \alpha}$ -regularity and conclusion. In this last step, we wish to prove an excess decay estimate for the Campanato-type excess

$$\Psi(\rho) := \rho^{2n-m} \int_{B_\rho(y)} |D^{n-1} A - (D^{n-1} A)_{y, \rho}|^2 dx,$$

for some $y \in B_1$ and $\rho < 1 - |y|$. To this end, we use the results from the preceding step, which imply in particular

$$\sum_{j=0}^{n-2} \int_{B_r} |D^j A|^{\frac{2n}{j+1}} dx \leq \widehat{C} r^m \quad \text{and} \quad r^{2n-m} \int_{B_r} |D^{n-1} A|^2 dx \leq \widehat{C} r^{2\alpha}$$

for any $\alpha \in (n-1, n)$ and any ball $B_r = B_r(y) \subset B_R$, with constants that might depend on A and R . In what follows, we fix a value $\alpha \in (\frac{n^2}{n+1}, n)$. Combining the preceding estimate with the bounds (6.4) for P_k , we infer similarly as in (7.9)

$$\begin{aligned} \left(r^{2n-m} \int_{B_r} |P_k[A]|^{\frac{2n}{k}} dx \right)^{\frac{k}{n}} & \leq C \left(r^{2n} + \sum_{j=0}^{n-1} r^{2n-m} \int_{B_r} |D^j A|^{\frac{2n}{j+1}} dx \right)^{\frac{k}{n}} \\ & \leq \widehat{C} r^{2k} + \widehat{C} r^{2\alpha \frac{k}{n}} \leq \widehat{C} r^{2\alpha \frac{n+1}{n}} \end{aligned} \quad (7.11)$$

for every $k \in \{n+1, \dots, 2n-1\}$. We again consider the decomposition $A = A_0 + A_1$ into a Δ^{n-1} -polyharmonic function A_1 and a function $A_0 \in W_0^{n-1, 2}(B_r)$.

Combining (7.11) with the bound (7.3), we deduce

$$\begin{aligned} r^{2n-m} \int_{B_r} |D^{n-1} A_0|^2 dx \\ \leq C \sum_{k+1}^{2n-1} \left(r^{2n-m} \int_{B_r} |P_k[A]|^{\frac{2n}{k}} dx \right)^{\frac{k}{n}} \leq \widehat{C} r^{2\alpha \frac{n+1}{n}}. \end{aligned} \quad (7.12)$$

Our next goal is an improvement of the excess decay estimate (7.4) for the polyharmonic function A_1 . For this aim, we observe that with $A_1 \in C^\infty(B_r)$, also $D^n A_1$ is polyharmonic, and thus we can estimate, following the lines of [GS, Lemma 6.2],

$$\begin{aligned} \rho^2 \int_{B_\rho} |D^n A_1|^2 dx &\leq C \rho^2 \|D^n A_1\|_{L^\infty(B_{r/2})}^2 \\ &\leq C \rho^2 \int_{B_r} |D^n A_1|^2 dx \\ &\leq C \left(\frac{\rho}{r} \right)^2 \int_{B_r} |D^{n-1} A_1 - (D^{n-1} A)_{B_r}|^2 dx \end{aligned}$$

for all $\rho \in (0, \frac{r}{2})$. Combining the above estimate with Poincaré's inequality, we infer

$$\int_{B_\rho} |D^{n-1} A_1 - (D^{n-1} A_1)_{B_\rho}|^2 dx \leq C \left(\frac{\rho}{r} \right)^2 \int_{B_r} |D^{n-1} A_1 - (D^{n-1} A)_{B_r}|^2 dx.$$

Next, we transfer this decay estimate to A by means of (7.12) as follows.

$$\begin{aligned} \Psi(\rho) &\leq \rho^{2n-m} \int_{B_\rho} |D^{n-1} A - (D^{n-1} A_1)_{B_\rho}|^2 dx \\ &\leq 2\rho^{2n-m} \int_{B_\rho} |D^{n-1} A_1 - (D^{n-1} A_1)_{B_\rho}|^2 dx + 2\rho^{2n-m} \int_{B_\rho} |D^{n-1} A_0|^2 dx \\ &\leq C \left(\frac{\rho}{r} \right)^{2n+2} r^{2n-m} \int_{B_r} |D^{n-1} A_1 - (D^{n-1} A)_{B_r}|^2 dx \\ &\quad + 2\rho^{2n-m} \int_{B_\rho} |D^{n-1} A_0|^2 dx \\ &\leq C \left(\frac{\rho}{r} \right)^{2n+2} \Psi(r) + C r^{2n-m} \int_{B_r} |D^{n-1} A_0|^2 dx \\ &\leq C \left(\frac{\rho}{r} \right)^{2n+2} \Psi(r) + \widehat{C} r^{2\alpha \frac{n+1}{n}}. \end{aligned}$$

Since by our choice of α , there holds $2\alpha \frac{n+1}{n} \in (2n, 2n+2)$, we can iterate the above estimate to get

$$\Psi(\rho) \leq \widehat{C} \rho^{2\alpha \frac{n+1}{n}} = \widehat{C} \rho^{2n+2\gamma}$$

for some $\gamma > 0$ and every $0 < \rho \leq r \leq R - |y|$. But this implies $A \in C_{\text{loc}}^{n-1, \gamma}(B_1)$ by Campanato's integral characterization of Hölder continuous functions.

Having arrived at this stage, the claim $A \in C^\infty(B_1)$ follows from classical Schauder theory, see e.g. [DN, Thm. 2'], which concludes the proof of the theorem. \square

8 Existence of minimizers

In this part of the paper, we formulate our existence theorems for both minimizers of Y_n and Z_n . In the proofs, however, we mention only Y_n . The modifications for Z_n are straightforward.

8.1 The critical dimension

In the critical dimension, we encounter the problem of a possible bubbling phenomenon during the minimizing procedure that might result in a change of the underlying bundle. However, certain topological invariants are preserved. Our considerations involve prescribing certain Chern classes of the bundle. The relations between Chern Classes and Sobolev maps with finite L^n -norm of the curvature have been explored by Uhlenbeck [Uh3], and for a minimizing procedure similar to ours by Sedlacek [Se].

In order to demonstrate how certain invariants are preserved, we restrict ourselves to the case of principal $SU(k)$ -bundles P over M . We recall that in this case, the Chern classes of the associated vector bundle $P_{\mathbb{C}^k} := P \times_\rho \mathbb{C}^k$ is given by

$$c_j(P_{\mathbb{C}^k}) = [p_j(\frac{i}{2\pi}F_A)] \in H_{\text{dR}}^{2j}(M),$$

Here, p_j denotes the j -th elementary symmetric polynomial of the eigenvalues and F_A is the curvature of any connection on P . Moreover, with $P \times_\rho \mathbb{C}^k$ we abbreviate the complex vector bundle associated to P by the representation $\rho : SU(k) \rightarrow GL(k, \mathbb{C})$ of $SU(k)$ on \mathbb{C}^k .

In order to compare the Chern classes of bundles over $M \setminus \Sigma$, the following lemma is crucial.

Lemma 8.1 *Assume that M is a compact manifold of dimension $m \geq 4$ and $\Sigma \subset M$ is a finite set. Then, the inclusion $\iota_0 : M \setminus \Sigma \rightarrow M$ induces an isomorphism*

$$\iota_0^* : H_{\text{dR}}^\ell(M) \rightarrow H_{\text{dR}}^\ell(M \setminus \Sigma)$$

for every $\ell \in \{2, \dots, m-2\}$. Similarly, if B is the union of finitely many, pairwise disjoint closed balls, then the inclusion $\iota_1 : M \setminus B \rightarrow M$ induces an isomorphism

$$\iota_1^* : H_{\text{dR}}^\ell(M) \rightarrow H_{\text{dR}}^\ell(M \setminus B).$$

Proof. For any finite set $\Sigma \subset M$ we may choose a union of finitely many, pairwise disjoint closed balls $B \supset \Sigma$. Since $M \setminus B$ is a deformation retract of $M \setminus \Sigma$, it suffices to prove the second statement of the lemma. To this end, we choose a

union of finitely many, pairwise disjoint *open* balls $\widehat{B} \supset B$. The Mayer-Vietoris sequence for the open sets $M \setminus B$ and \widehat{B} reads

$$\dots \rightarrow H_{\text{dR}}^{\ell-1}(\widehat{B} \setminus B) \rightarrow H_{\text{dR}}^{\ell}(M) \xrightarrow{(\iota_1^*, \iota_2^*)} H_{\text{dR}}^{\ell}(M \setminus B) \oplus H_{\text{dR}}^{\ell}(\widehat{B}) \rightarrow H_{\text{dR}}^{\ell}(\widehat{B} \setminus B) \rightarrow \dots$$

Here, $\widehat{B} \setminus B$ is homotopy equivalent to N spheres of dimension $m - 1 > \ell$ and \widehat{B} is the union of N balls of dimension $m > \ell$. Therefore, we have

$$H_{\text{dR}}^{\ell}(\widehat{B}) = 0 = H_{\text{dR}}^{\ell-1}(\widehat{B} \setminus B) = H_{\text{dR}}^{\ell}(\widehat{B} \setminus B)$$

for all $\ell \in \{2, \dots, m - 2\}$. Plugging this into the Mayer-Vietoris sequence stated above, we infer that $\iota_1^* : H_{\text{dR}}^{\ell}(M) \rightarrow H_{\text{dR}}^{\ell}(M \setminus B)$ is an isomorphism. \square

The following lemma will ensure that the Chern classes are preserved under weak L^p -convergence. The proof is a slight modification of the arguments in [ISS, Cor. 5.2].

Lemma 8.2 *Let M be a regular open subset of a smooth compact manifold, $k \in \mathbb{N}$ and $p > 1$. We consider a k -form $\phi_0 \in C^{\infty}(\overline{M}, \wedge^k T^*M)$ and a sequence $\phi_i \in [\phi_0] \cap L^p(M, \wedge^k T^*M)$ with $\phi_i \rightarrow \phi \in C^{\infty}(M, \wedge^k T^*M)$ weakly in L^p . Then $\phi \in [\phi_0]$.*

Proof. Since $\phi_i \in [\phi_0]$, there are smooth $(k - 1)$ -forms ω_i with

$$d\omega_i = \phi_i - \phi_0 \rightarrow \phi - \phi_0 \quad \text{weakly in } L^p, \text{ as } i \rightarrow \infty.$$

By an approximation argument, we may assume $\omega_i \in C^{\infty}(\overline{M}, \wedge^{k-1} T^*M)$ for all $i \in \mathbb{N}$. Following the strategy in [ISS, Cor. 5.2], we can find a $(k - 1)$ -form $\omega \in W^{1,p}(M, \wedge^{k-1} T^*M)$ with $d\omega_i \rightarrow d\omega$ weakly in L^p as $i \rightarrow \infty$, and in particular

$$d\omega = \phi - \phi_0.$$

Moreover, in [ISS, Cor. 5.2] the $(k - 1)$ -form ω is constructed as the weak limit of coexact forms in $W^{1,p}(M, \wedge^{k-1} T^*M)$, which implies $d^* \omega = 0$. We deduce

$$\Delta \omega = d^* d\omega = d^*(\phi - \phi_0) \in C^{\infty}(M, \wedge^{k-1} T^*M),$$

and elliptic regularity theory yields $\omega \in C^{\infty}(M, \wedge^{k-1} T^*M)$. We thereby arrive at $\phi = \phi_0 + d\omega \in [\phi_0]$, which completes the proof of the lemma. \square

For the formulation of the minimization problem that we wish to solve, we fix a reference bundle P_0 over a manifold M of dimension $m = 2n$. We want to prescribe the Chern classes $c_j^0 := c_j((P_0)_{\mathbb{C}^k}) \in H^{2j}(M)$ for $j \in \{1, \dots, n - 1\}$. More precisely, we define the class of admissible bundles by

$$\mathcal{P}_*(M, P_0) := \{P \in \mathcal{P}(M) : c_j(P_{\mathbb{C}^k}) = c_j^0 \text{ for } j = 1, \dots, n - 1\},$$

where $\mathcal{P}(M)$ denotes the class of smooth principal $SU(k)$ -bundles over M . In order to emphasize the occurrence of different bundles during the minimization process, we will frequently use the notation

$$Y_n(A, P) := Y_n(A)$$

if A is a smooth connection on the bundle $P \in \mathcal{P}_*(M, P_0)$.

Having introduced the setting, we are ready to state our existence result in the critical case.

Theorem 8.3 *Let M be a compact manifold of dimension $m = 2n$, and A_0 a smooth reference connection on a principal $SU(k)$ -bundle P_0 over M as above. Then there is a principal $SU(k)$ -bundle $P \in \mathcal{P}_*(M, P_0)$ and a smooth connection A on P that minimizes the functional Y_n (or Z_n) in the class of smooth connections on bundles in $\mathcal{P}_*(M, P_0)$.*

Proof. The proof is divided into several steps. We start with

Step 1: Convergence in Uhlenbeck gauges. We choose a minimizing sequence of smooth connections A_i on bundles $P_i \in \mathcal{P}_*(M, P_0)$ for the functional Y_n . From Theorem 3.1 we deduce

$$\sup_{i \in \mathbb{N}} \|D_{A_i}^\ell F_{A_i}\|_{L^{2n/(\ell+2)}(M)} < \infty \quad \text{for } \ell = 0, \dots, n-2. \quad (8.1)$$

Writing μ_M for the Riemannian measure on M , we define a sequence of Radon measures on M by

$$\mu_i := \mu_M \llcorner (|d_{A_i}^{*\wedge n-2} F_{A_i}|^2 + |F_{A_i}|^n).$$

The minimizing property of A_i implies

$$\sup_{i \in \mathbb{N}} \mu_i(M) = \sup_{i \in \mathbb{N}} Y_n(A_i, P_i) < \infty.$$

Therefore, we can find a Radon measure μ on M such that $\mu_i \xrightarrow{*} \mu$ weakly* in the space of Radon measures, possibly after extracting a subsequence. With the constant $\kappa > 0$ from Theorem 4.1, we define the singular set of the limit bundle by

$$\Sigma := \{x \in M : \mu(\{x\}) \geq \kappa\}$$

and let $N := \# \Sigma$. From the definition of Σ and μ , it is evident that for every $x \in M \setminus \Sigma$, we can find a ball $U \ni x$ such that

$$\limsup_{i \rightarrow \infty} \mu_i(2U) \leq \mu(\overline{2U}) < \kappa. \quad (8.2)$$

Here and in what follows, we use the notation $2U$ for the ball with double radius and the same center as U . We choose a countable cover U_α , $\alpha \in \mathbb{N}$, of balls with

the above property and $\cup_{\alpha \in \mathbb{N}} U_\alpha = M \setminus \Sigma$. In particular, the property (8.2) implies that for every $\alpha \in \mathbb{N}$ and every sufficiently large $i \geq i_0(\alpha)$, we have

$$\int_{2U_\alpha} |d_{A_i}^{*\wedge n-2} F_{A_i}|^2 + |F_{A_i}|^n dx < \kappa. \quad (8.3)$$

Therefore, the Gauge Theorem 4.1 provides us with gauge transformations $u_{i,\alpha} \in W^{n,2}(U_\alpha, G)$ such that $A_i^\alpha := u_{i,\alpha}^* A_i$ satisfies $d^* A_i^\alpha = 0$ on U_α and moreover

$$\begin{aligned} & \sup_{i \geq i_0(\alpha)} \sum_{\ell=0}^{n-1} \|D^\ell A_i^\alpha\|_{L^{2n/(\ell+1)}(U_\alpha)} \\ & \leq C \sup_{i \geq i_0(\alpha)} \left(\|D_{A_i}^{n-2} F_{A_i}\|_{L^2(2U_\alpha)} + \|F_{A_i}\|_{L^n(2U_\alpha)} \right) < \infty \end{aligned}$$

for every $\alpha \in \mathbb{N}$, where the finiteness of the right-hand side is a consequence of (8.1). By extraction of a subsequence of i (possibly depending on α), we can thus achieve the convergence to a local limit connection $A^\alpha \in W^{n-1,2}(U_\alpha, \wedge^1 \mathbb{R}^m \otimes \mathfrak{su}(k))$ in the sense

$$\begin{cases} A_i^\alpha \rightharpoonup A^\alpha & \text{weakly in } W^{n-1,2}(U_\alpha), \\ D^\ell A_i^\alpha \rightarrow D^\ell A^\alpha & \text{strongly in } L^p(U_\alpha) \text{ and a.e.} \\ & \text{for all } \ell \leq n-2 \text{ and } p < \frac{2n}{\ell+1}. \end{cases} \quad (8.4)$$

In particular, the limit connection is still in Uhlenbeck gauge, i. e. we have

$$d^* A^\alpha = 0 \quad \text{on } U_\alpha. \quad (8.5)$$

For the curvature of the connection, the above convergence implies

$$\begin{cases} d_{A_i^\alpha}^{*\wedge n-2} F_{A_i^\alpha} \rightharpoonup d_{A^\alpha}^{*\wedge n-2} F_{A^\alpha} & \text{weakly in } L^2(U_\alpha), \\ D_{A_i^\alpha}^\ell F_{A_i^\alpha} \rightharpoonup D_{A^\alpha}^\ell F_{A^\alpha} & \text{weakly in } L^{2n/(\ell+2)}(U_\alpha) \quad \forall \ell \leq n-2, \\ F_{A_i^\alpha} \rightarrow F_{A^\alpha} & \text{strongly in } L^p(U_\alpha) \quad \forall p < n \\ & \text{and a.e., provided } n > 2, \end{cases} \quad (8.6)$$

as $i \rightarrow \infty$. The gauge transformations $u_{i,\alpha}$ define transition functions $\phi_{\alpha\beta}^i \in W^{n,2}(U_\alpha \cap U_\beta, SU(k))$ by the identity

$$u_{i,\beta} = u_{i,\alpha} \phi_{\alpha\beta}^i \quad \text{on } U_\alpha \cap U_\beta,$$

for all $i \in \mathbb{N}$ and all $\alpha, \beta \in \mathbb{N}$ for which $U_\alpha \cap U_\beta \neq \emptyset$. From the transformation rule for connections, we have

$$d\phi_{\alpha\beta}^i = \phi_{\alpha\beta}^i A_i^\beta - A_i^\alpha \phi_{\alpha\beta}^i \quad \text{on } U_\alpha \cap U_\beta. \quad (8.7)$$

Using the fact that A_i^α and A_i^β are both bounded sequences in $W^{n-1,2}$ and $SU(k)$ is compact, we inductively deduce from (8.7) that

$$\sup_{i \in \mathbb{N}} \|D^\ell \phi_{\alpha\beta}^i\|_{L^{2n/\ell+1}} < \infty$$

for all $\ell \in \{0, \dots, n\}$. Therefore, we can achieve the convergence to maps $\phi_{\alpha\beta} \in W^{n,2}(U_\alpha \cap U_\beta, SU(k))$ in the sense

$$\begin{cases} \phi_{\alpha\beta}^i \rightharpoonup \phi_{\alpha\beta} & \text{weakly in } W^{n,2}(U_\alpha \cap U_\beta, SU(k)) \\ D^\ell \phi_{\alpha\beta}^i \rightarrow D^\ell \phi_{\alpha\beta} & \text{almost everywhere } \forall \ell \leq n-1, \end{cases} \quad (8.8)$$

as $i \rightarrow \infty$. From the almost everywhere convergence of the $\phi_{\alpha\beta}^i$ and A_i^α by (8.4) and (8.8), we infer that we can pass to the limit in (8.7), with the result

$$d\phi_{\alpha\beta} = \phi_{\alpha\beta} A^\beta - A^\alpha \phi_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta. \quad (8.9)$$

Moreover, the almost everywhere convergence guarantees that the cocycle conditions

$$\phi_{\alpha\beta} \phi_{\beta\gamma} = \phi_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \quad (8.10)$$

are preserved in the limit. We note that if the transition functions $\{\phi_{\alpha\beta}\}_{\alpha,\beta \in \mathbb{N}}$ were of class C^∞ , they would define a new principal bundle P over $\cup_{\alpha \in \mathbb{N}} U_\alpha = M \setminus \Sigma$. Therefore we turn our attention to

Step 2: Regularity of the limit configuration. The smoothness of the local limit connections A^α will follow from Theorem 7.1 once we have established that A^α weakly solves the Euler-Lagrange equations (6.2) on U_α . For this it suffices to prove

$$Y_n(A^\alpha) \leq Y_n(A^\alpha + \varphi) \quad (8.11)$$

for all $\varphi \in C_{\text{cpt}}^\infty(U_\alpha, \wedge^1 \mathbb{R}^m \otimes \mathfrak{su}(k))$. If this was not the case, we could find $\varphi \in C_{\text{cpt}}^\infty(U_\alpha, \wedge^1 \mathbb{R}^m \otimes \mathfrak{su}(k))$ for which the inequality (8.11) does not hold. The main step to reach a contradiction is to prove the

Claim. As $i \rightarrow \infty$, we have the convergence

$$Y_n(A_i^\alpha + \varphi) - Y_n(A_i^\alpha) \longrightarrow Y_n(A^\alpha + \varphi) - Y_n(A^\alpha) < 0. \quad (8.12)$$

For the **proof of the claim**, we begin by calculating

$$F_{A_i^\alpha + \varphi} = F_{A_i^\alpha} + [A_i^\alpha, \varphi] + d\varphi + \frac{1}{2}[\varphi, \varphi],$$

and the same holds for A^α instead of A_i^α . As a consequence of the strong convergence $A_i^\alpha \rightarrow A^\alpha$ in L^n according to (8.4) and $\varphi \in C_{\text{cpt}}^\infty$, we deduce

$$F_{A_i^\alpha + \varphi} - F_{A_i^\alpha} \longrightarrow F_{A^\alpha + \varphi} - F_{A^\alpha} \quad \text{strongly in } L^n, \text{ as } i \rightarrow \infty. \quad (8.13)$$

Abbreviating

$$I(\omega, \psi) := \int_0^1 n |\omega + t(\psi - \omega)|^{n-2} (\omega + t(\psi - \omega)) dt$$

for $\omega, \psi \in \wedge^2 \mathbb{R}^m \otimes \mathfrak{su}(k)$ we have $|\omega|^n - |\psi|^n = I(\omega, \psi) \cdot (\omega - \psi)$. As $i \rightarrow \infty$, we moreover have

$$I(F_{A_i^\alpha + \varphi}, F_{A_i^\alpha}) \rightharpoonup I(F_{A^\alpha + \varphi}, F_{A^\alpha}) \quad \text{weakly in } L^{\frac{n}{n-1}}(U_\alpha, \wedge^2 \otimes \mathfrak{su}(k)).$$

In fact, in the case $n > 2$, this convergence holds almost everywhere by (8.6) and the weak convergence follows since $I(F_{A_i^\alpha + \varphi}, F_{A_i^\alpha})$ is bounded in $L^{n/(n-1)}$ because of $|I(F_{A_i^\alpha + \varphi}, F_{A_i^\alpha})| \leq n(|F_{A_i^\alpha + \varphi}| + |F_{A_i^\alpha}|)^{n-1}$. For $n = 2$, however, the term $I(F_{A_i^\alpha + \varphi}, F_{A_i^\alpha})$ depends linearly on the curvature and therefore, the claim follows from the weak convergence in (8.6). Combining the above results, we infer

$$\begin{aligned} \int_{U_\alpha} |F_{A_i^\alpha + \varphi}|^n - |F_{A_i^\alpha}|^n dx &= \int_{U_\alpha} I(F_{A_i^\alpha + \varphi}, F_{A_i^\alpha}) \cdot (F_{A_i^\alpha + \varphi} - F_{A_i^\alpha}) dx \\ &\xrightarrow{i \rightarrow \infty} \int_{U_\alpha} I(F_{A^\alpha + \varphi}, F_{A^\alpha}) \cdot (F_{A^\alpha + \varphi} - F_{A^\alpha}) dx \\ &= \int_{U_\alpha} |F_{A^\alpha + \varphi}|^n - |F_{A^\alpha}|^n dx. \end{aligned} \quad (8.14)$$

Similarly, we proceed for the derivatives of the curvature. Inductively, we calculate for every $\ell \in \{1, \dots, n-2\}$

$$\begin{aligned} d_{A_i^\alpha + \varphi}^{*\wedge \ell} F_{A_i^\alpha + \varphi} - d_{A_i^\alpha}^{*\wedge \ell} F_{A_i^\alpha} \\ = \sum_j \sum_{k=1}^K D^{j_1-1} \varphi \# \dots D^{j_k-1} \varphi \# D^{j_{k+1}-1} A_i^\alpha \# \dots D^{j_K-1} A_i^\alpha, \end{aligned} \quad (8.15)$$

where the sum is taken over all tuples $j = (j_1, \dots, j_K)$ of $K \in \mathbb{N}$ naturals with $j_1 + \dots + j_K \leq \ell + 2$. We note that at least one of the factors in each term of the above sum contains φ and therefore is bounded. Moreover, we note that for every partition j occurring in the above sum for $\ell = n-2$, we have $\frac{j_{k+1}}{2n} + \dots + \frac{j_K}{2n} < \frac{1}{2}$. Because of the strong convergence (8.4)₂, we thereby infer

$$d_{A_i^\alpha + \varphi}^{*\wedge n-2} F_{A_i^\alpha + \varphi} - d_{A_i^\alpha}^{*\wedge n-2} F_{A_i^\alpha} \longrightarrow d_{A^\alpha + \varphi}^{*\wedge n-2} F_{A^\alpha + \varphi} - d_{A^\alpha}^{*\wedge n-2} F_{A^\alpha} \quad \text{in } L^2, \quad (8.16)$$

as $i \rightarrow \infty$. Joining this with the weak convergence in (8.6), we deduce

$$\begin{aligned} \int_{U_\alpha} |d_{A_i^\alpha + \varphi}^{*\wedge n-2} F_{A_i^\alpha + \varphi}|^2 - |d_{A_i^\alpha}^{*\wedge n-2} F_{A_i^\alpha}|^2 dx \\ = \int_{U_\alpha} (d_{A_i^\alpha + \varphi}^{*\wedge n-2} F_{A_i^\alpha + \varphi} + d_{A_i^\alpha}^{*\wedge n-2} F_{A_i^\alpha}) \cdot (d_{A_i^\alpha + \varphi}^{*\wedge n-2} F_{A_i^\alpha + \varphi} - d_{A_i^\alpha}^{*\wedge n-2} F_{A_i^\alpha}) dx \\ \xrightarrow{i \rightarrow \infty} \int_{U_\alpha} (d_{A^\alpha + \varphi}^{*\wedge n-2} F_{A^\alpha + \varphi} + d_{A^\alpha}^{*\wedge n-2} F_{A^\alpha}) \cdot (d_{A^\alpha + \varphi}^{*\wedge n-2} F_{A^\alpha + \varphi} - d_{A^\alpha}^{*\wedge n-2} F_{A^\alpha}) dx \\ = \int_{U_\alpha} |d_{A^\alpha + \varphi}^{*\wedge n-2} F_{A^\alpha + \varphi}|^2 - |d_{A^\alpha}^{*\wedge n-2} F_{A^\alpha}|^2 dx. \end{aligned} \quad (8.17)$$

We can combine the convergences (8.14) and (8.17) to yield the claim (8.12). Now that the claim is proven, we resume Step 2. We define new connections \tilde{A}_i on the bundles P_i in such a way that they agree with A_i on $P_i|_{M \setminus U_\alpha}$ and such that $u_{i,\alpha}^* \tilde{A}_i = A_i^\alpha + \varphi$ on $P_i|_{U_\alpha}$. In view of (8.12), we have

$$\begin{aligned} Y_n(\tilde{A}_i, P_i) - Y_n(A_i, P_i) &= Y_n(A_i^\alpha + \varphi) - Y_n(A_i^\alpha) \\ &\xrightarrow{i \rightarrow \infty} Y_n(A^\alpha + \varphi) - Y_n(A^\alpha) < 0 \end{aligned}$$

by the choice of φ as a counterexample to (8.11). Therefore, in both cases we infer

$$\lim_{i \rightarrow \infty} Y_n(\tilde{A}_i, P_i) < \lim_{i \rightarrow \infty} Y_n(A_i, P_i) = \inf_{(A, P)} Y_n(A, P),$$

where the infimum is taken over all $P \in \mathcal{P}_*(M, P_0)$ and all smooth connections A on P . We have thus reached the contradiction that proves (8.11). Lemma 6.3 now yields that A^α is a weak solution of (6.2), and since it is moreover in Uhlenbeck gauge according to (8.5), we infer from Theorem 7.1 that $A^\alpha \in C^\infty(U_\alpha, \wedge^1 \mathbb{R}^m \otimes \mathfrak{su}(k))$. Furthermore, from the transition formula (8.9) we inductively deduce $|D^\ell \phi_{\alpha\beta}| \in L_{\text{loc}}^\infty(U_\alpha \cap U_\beta)$ for all $\ell \in \mathbb{N}$ and thereby $\phi_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, SU(k))$. Keeping in mind the cocycle conditions (8.10), we can construct a new principal bundle P over $\bigcup_{\alpha \in \mathbb{N}} U_\alpha = M \setminus \Sigma$ with the transition functions $\{\phi_{\alpha\beta}\}$. From (8.9), we moreover know

$$A^\beta = \phi_{\alpha\beta}^{-1} A^\alpha \phi_{\alpha\beta} + \phi_{\alpha\beta}^{-1} d\phi_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta$$

for all $\alpha, \beta \in \mathbb{N}$ for which the latter set is not empty. This means that the collection $\{A^\alpha\}$ defines a connection A on the new bundle P over $M \setminus \Sigma$.

Step 3: Removability of the point singularities

We write $\Sigma = \{x_1, \dots, x_N\}$ for the singular set of the bundle P and choose pairwise disjoint open balls $V_\ell \subset M$ with centers in the points $x_\ell \in \Sigma$ for $\ell \in \mathbb{N}$. Then, we can find a finite subset $I \subset \mathbb{N}$ with

$$M = \bigcup_{\ell=1}^N V_\ell \cup \bigcup_{\alpha \in I} U_\alpha. \quad (8.18)$$

As a consequence of (8.1), the lower semicontinuity of the norm with respect to weak convergence and the gauge invariance of $D_A^\ell F_A$, we have

$$\sum_{\ell=0}^{n-2} \|D_A^\ell F_A\|_{L^{2n/(\ell+2)}(M)} < \infty.$$

We may thus apply the Removable Singularity Theorem 5.2 on each ball $V_\ell \setminus \{x_\ell\}$. This theorem provides us with local $W_{\text{loc}}^{n,2}$ -trivializations $v_\ell : \pi^{-1}(V_\ell) \rightarrow (V_\ell \setminus \{x_\ell\}) \times SU(k)$ of the bundle $\pi : P \rightarrow M \setminus \Sigma$ and $\hat{A}_\ell \in W^{n-1,2}(V_\ell, \wedge^1 \mathbb{R}^m \otimes \mathfrak{su}(k))$ such that

$$(v_\ell)_* A = \hat{A}_\ell|_{V_\ell \setminus \{x_\ell\}} \quad \text{for } \ell = 1, \dots, N.$$

These trivializations give rise to the transition functions $\psi_{\alpha\ell} \in W^{n,2}(V_\ell \cap U_\alpha, SU(k))$ defined by

$$v_\ell = u_\alpha \psi_{\alpha\ell} \quad \text{on } V_\ell \cap U_\alpha,$$

for any $\ell \in \{1, \dots, N\}$ and $\alpha \in I$ for which $V_\ell \cap U_\alpha \neq \emptyset$. By diminishing the balls V_ℓ is necessary, we can achieve that the smallness assumption from the

Gauge Theorem 4.1 is satisfied on $2V_\ell$. Therefore, we may assume that the trivializations v_ℓ have been chosen in such a way that $d^*\hat{A}_\ell = 0$ holds for every $\ell \in \{1, \dots, N\}$. The arguments from Step 2 of the proof therefore imply that \hat{A}_ℓ weakly solves an Euler-Lagrange equation of the form (6.3) on $V_\ell \setminus \{x_\ell\}$. A standard capacity argument then shows that \hat{A}_ℓ actually solves (6.3) on the whole ball V_ℓ . We are thus in a position to apply the Regularity Theorem 7.1 to deduce that $\hat{A}_\ell \in C^\infty(V_\ell, \wedge^1 \mathbb{R}^m \otimes \mathfrak{su}(k))$. In the same manner as in Step 2 we then infer that the transition functions $\psi_{\alpha\ell}$ are smooth. This implies that the collection of transition functions $\{\phi_{\alpha\beta}\} \cup \{\psi_{\alpha\ell}\}$ relative to the open cover (8.18) defines a smooth bundle \hat{P} over M , and the collection $\{A_\alpha\} \cup \{\hat{A}_\ell\}$ of local connection forms defines a smooth connection \hat{A} on \hat{P} .

Step 4: The Chern classes are preserved. Next we wish to prove that \hat{P} possesses the same Chern classes as the reference bundle P_0 and therefore, $\hat{P} \in \mathcal{P}_*(M, P_0)$ is an admissible comparison bundle. Since the inclusion $\iota : M \setminus \Sigma \rightarrow M$ induces an isomorphism $\iota^* : H_{\text{dR}}^{2j}(M) \rightarrow H_{\text{dR}}^{2j}(M \setminus \Sigma)$ by Lemma 8.1, it suffices to show that

$$c_j(P_{C^k}) = \iota^* c_j^0 \in H_{\text{dR}}^{2j}(M \setminus \Sigma) \quad \text{for } j = 1, \dots, n-1. \quad (8.19)$$

We choose a union of N pairwise disjoint closed balls with centers in the points of Σ . Since $M \setminus B$ is a deformation retract of $M \setminus \Sigma$, it suffices to show

$$p_j\left(\frac{i}{2\pi}FA\Big|_{M \setminus B}\right) \in \iota_B^* c_j^0 \in H_{\text{dR}}^{2j}(M \setminus B), \quad (8.20)$$

where $\iota_B : M \setminus B \hookrightarrow M$ is the inclusion. Because $\overline{M \setminus B}$ is compact, it is covered by finitely many of the open sets U_α . Since the A_i are smooth connections on principal bundles $P_i \in \mathcal{P}_*(M, P_0)$, of which we prescribed the corresponding Chern classes, we know

$$p_j\left(\frac{i}{2\pi}FA_i\right) \in c_j^0 \in H_{\text{dR}}^{2j}(M),$$

for all $i \in \mathbb{N}$ and $j = 1, \dots, n-1$. Now we apply Lemma 8.1, which states that ι_B induces an isomorphism on the cohomology groups of order $2j$ for $j = 1, \dots, n-1$. The preceding formula thereby implies

$$p_j\left(\frac{i}{2\pi}FA_i\Big|_{M \setminus B}\right) \in \iota_B^* c_j^0 \in H_{\text{dR}}^{2j}(M \setminus B). \quad (8.21)$$

In order to prove (8.20), it therefore suffices to check that this property is preserved in the limit $i \rightarrow \infty$. We first consider the case $n > 2$, in which the strong convergence in (8.6) implies strong convergence $F_{A_i^\alpha} \rightarrow F_{A^\alpha}$ of the local representations of F_{A_i} in L^p -norm for every $p < n$. Keeping in mind that p_j is a polynomial of order j and using gauge invariance, we deduce

$$p_j\left(\frac{i}{2\pi}FA_i\Big|_{M \setminus B}\right) \rightarrow p_j\left(\frac{i}{2\pi}FA\Big|_{M \setminus B}\right) \quad \text{in } L^p(M \setminus B, \wedge^{2j} T^*M) \quad \forall 1 < p < \frac{n}{j}$$

as $i \rightarrow \infty$, provided $1 \leq j \leq n-1$ and $n > 2$. In view of Lemma 8.2, this convergence and (8.21) imply the claim (8.20).

Finally, in the case of the Yang-Mills equation $n = 2$, the weak convergence $F_{A_i^\alpha} \rightharpoonup F_{A^\alpha}$ in L^2 and the linearity of p_1 imply

$$p_1\left(\frac{i}{2\pi}F_{A_i}\big|_{M\setminus B}\right) \rightharpoonup p_1\left(\frac{i}{2\pi}F_A\big|_{M\setminus B}\right) \quad \text{weakly in } L^2(M \setminus B, \wedge^2 T^*M)$$

as $i \rightarrow \infty$. This implies by Lemma 8.2 that the first Chern class is preserved in the limit, which is the only assertion claimed in (8.20) for $n = 2$. We have thereby established the claim (8.19) and conclude that the limit bundle \widehat{P} possesses the desired Chern classes, which means $\widehat{P} \in \mathcal{P}_*(M, P_0)$.

Final step: Minimization property of the limit connection. We choose a partition of unity $\{\zeta_\alpha\}_{\alpha \in \mathbb{N}}$ subordinate to the cover U_α of $M \setminus \Sigma$. From (8.6) and the lower semi-continuity of the norm with respect to weak convergence, we infer

$$\begin{aligned} \int \zeta_\alpha (|d_A^{*\wedge n-2} F_A|^2 + |F_A|^n) dx &\leq \lim_{i \rightarrow \infty} \int \zeta_\alpha (|d_{A_i}^{*\wedge n-2} F_{A_i}|^2 + |F_{A_i}|^n) dx \\ &= \lim_{i \rightarrow \infty} \int \zeta_\alpha d\mu_i = \int \zeta_\alpha d\mu, \end{aligned}$$

where we used the weak*-convergence $\mu_i \xrightarrow{*} \mu$ in the last step. Summing over $\alpha \in \mathbb{N}$ and using the fact that ζ_α is a partition of unity on $M \setminus \Sigma$, we arrive at

$$\begin{aligned} Y_n(\widehat{A}, \widehat{P}) &= Y_n(A, P) \leq \mu(M \setminus \Sigma) \leq \mu(M) \\ &= \lim_{i \rightarrow \infty} \mu_i(M) = \lim_{i \rightarrow \infty} Y_n(A_i, P_i) = \inf Y_n. \end{aligned}$$

This implies that the pair $(\widehat{A}, \widehat{P})$ minimizes the functional Y_n in the class of principal bundles $P \in \mathcal{P}_*(M, P_0)$ and of smooth connections A on P . \square

8.2 The subcritical dimensions

If $m < 2n$, we can even minimize Y_n in an arbitrary fixed bundle over M :

Theorem 8.4 *Let M be a compact manifold of dimension $m < 2n$ and P a smooth principal G -bundle over M with compact structure group G . Then there is a smooth connection A minimizing Y_n (or Z_n) among all connections on P of class $W^{n-1,2}$.*

In order to show that the bundle does not change when passing to the limit in a minimizing sequence, we need the following patching construction that goes back to Uhlenbeck [Uh2, Prop. 3.2]. A detailed proof can be found in [We, Lemma 7.2(i)], where the result is even proven for transition functions $g_{\alpha\beta}, h_{\alpha\beta}$ that are only of class $W^{k+1,p}$ for any $k \in \mathbb{N}$ and $p > \frac{m}{2}$, with resulting gauge transformations of the same class. Since we can choose $k \in \mathbb{N}$ arbitrarily, this result includes the case of C^∞ functions.

Lemma 8.5 Consider a locally finite open cover $M = \bigcup_{\alpha \in \mathbb{N}} U_\alpha$ of a compact manifold M by precompact sets U_α . There are a constant δ_M , only depending on the geometry of M , and open subsets $V_\alpha \subset U_\alpha$, $\alpha \in \mathbb{N}$, still covering M such that the following holds. For any two sets of transition functions $g_{\alpha\beta}, h_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, G)$, $\alpha, \beta \in \mathbb{N}$, that both satisfy the cocycle conditions

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{and} \quad h_{\alpha\beta}h_{\beta\gamma} = h_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

and furthermore

$$\sup_{x \in U_\alpha \cap U_\beta} d(g_{\alpha\beta}(x), h_{\alpha\beta}(x)) \leq \delta_M \quad \text{for all } \alpha, \beta \in \mathbb{N}, \quad (8.22)$$

there exist local gauge transformations $h_\alpha \in C^\infty(V_\alpha, G)$ for all $\alpha \in \mathbb{N}$ such that

$$h_\alpha^{-1} h_{\alpha\beta} h_\beta = g_{\alpha\beta} \quad \text{on } V_\alpha \cap V_\beta,$$

provided $V_\alpha \cap V_\beta \neq \emptyset$.

After these preparations, we can proceed to the

Proof of Theorem 8.4. We start by choosing a minimizing sequence of $W^{n-1,2}$ connections A_i , $i \in \mathbb{N}$ on the bundle P . Theorem 3.1 provides us with the bound

$$\sup_{i \in \mathbb{N}} \|D_{A_i}^\ell F_{A_i}\|_{L^{2n/(\ell+2)}(M)} < \infty \quad \text{for } \ell = 0, \dots, n-2. \quad (8.23)$$

Step 1: Construction of local Uhlenbeck gauges. Since we are in the subcritical dimension $m < 2n$, we can always choose a radius $r_0 > 0$ so small that

$$\sup_{i \in \mathbb{N}} r^{2n-m} \int_{B_r} |F_{A_i}|^n dx \leq r^{2n-m} \sup_{i \in \mathbb{N}} Y_n(A_i) < \kappa^n \quad (8.24)$$

holds for any ball $B_r \subset M$ of radius $r \leq r_0$, where here, we chose the constant $\kappa = \kappa(M) > 0$ from the Gauge Theorem 4.1. Now we cover M by finitely many balls U_α , $\alpha = 1, \dots, L$ of the same radius $r \leq \frac{1}{2}r_0$ in such a way that the bundle P is trivial over U_α for each $\alpha = 1, \dots, L$. This means that we can find bundle isomorphisms

$$U_\alpha \times G \rightarrow P|_{U_\alpha}, \quad (x, g) \mapsto g_\alpha(x)g$$

given by C^∞ -sections $g_\alpha : U_\alpha \rightarrow P$. The local trivializations give rise to transition functions $g_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, G)$ defined by

$$g_\beta = g_\alpha g_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta \quad (8.25)$$

for all α, β for which the intersection $U_\alpha \cap U_\beta$ is non-empty. From (8.24), we know that

$$\sup_{i \in \mathbb{N}} (2r)^{2n-m} \int_{2U_\alpha} |F_{A_i}|^n dx < \kappa^n. \quad (8.26)$$

The Gauge Theorem 4.1 thus yields gauge transformations $u_{i,\alpha} \in W^{n,2}(U_\alpha, G)$ such that the localized connection forms $A_i^\alpha := u_{i,\alpha}^* A_i$ satisfy $d^* A_i^\alpha = 0$ and

$$\begin{aligned} \sup_{i \in \mathbb{N}} \sum_{\ell=0}^{n-1} r^{\frac{2n-m}{2n}(\ell+1)} \|D^\ell A_i^\alpha\|_{L^{2n/(\ell+1)}(U_\alpha)} \\ \leq C \sup_{i \in \mathbb{N}} \left(r^{\frac{2n-m}{2}} \|D_{A_i}^{n-2} F_{A_i}\|_{L^2(2U_\alpha)} + r^{\frac{2n-m}{n}} \|F_{A_i}\|_{L^n(2U_\alpha)} \right) < \infty \end{aligned} \quad (8.27)$$

for every $\alpha = 1, \dots, L$, where the finiteness of the right-hand side follows from (8.23). The gauge transformations $u_{i,\alpha}$ define transition functions $\phi_{\alpha\beta}^i \in W^{n,2}(U_\alpha \cap U_\beta, G)$ by the identity

$$u_{i,\beta} = u_{i,\alpha} \phi_{\alpha\beta}^i \quad \text{on } U_\alpha \cap U_\beta.$$

From $A_i^\beta = (\phi_{\alpha\beta}^i)^* A_i^\alpha$ we infer

$$d\phi_{\alpha\beta}^i = \phi_{\alpha\beta}^i A_i^\beta - A_i^\alpha \phi_{\alpha\beta}^i \quad \text{on } U_\alpha \cap U_\beta.$$

Step 2: Convergence to a smooth limit connection. This step is analogous to the critical case in Theorem 8.3. Starting from the estimate (8.27), we can argue as in the critical case in order to find local limit connections $A_*^\alpha \in W^{n-1,2}(U_\alpha, \wedge^1 \mathbb{R}^m \otimes \mathfrak{g})$ and transition functions $\phi_{\alpha\beta} \in W^{n,2}(U_\alpha \cap U_\beta, G)$ such that after extraction of a subsequence we have

$$\begin{cases} A_i^\alpha \rightharpoonup A_*^\alpha & \text{weakly in } W^{n-1,2}(U_\alpha, \wedge^1 \mathbb{R}^m \otimes \mathfrak{g}), \\ \phi_{\alpha\beta}^i \rightharpoonup \phi_{\alpha\beta} & \text{weakly in } W^{n,2}(U_\alpha \cap U_\beta, G). \end{cases} \quad (8.28)$$

The weak convergence implies that the local limit connections are still in Uhlenbeck gauge, i. e.

$$d^* A_*^\alpha = 0 \quad \text{on } U_\alpha. \quad (8.29)$$

Moreover, the limit connections satisfy the transformation rule

$$d\phi_{\alpha\beta} = \phi_{\alpha\beta} A_*^\beta - A_*^\alpha \phi_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta, \quad (8.30)$$

and the cocycle conditions are also preserved in the limit in the sense

$$\phi_{\alpha\beta} \phi_{\beta\gamma} = \phi_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \quad (8.31)$$

As in step 2 of the proof of Theorem 8.3 we deduce that A_*^α satisfies the Euler-Lagrange system for Y_n . The proof of this fact becomes only easier in the subcritical case because now the compact embedding $W^{n-1,2} \hookrightarrow W^{\ell,2n/\ell+1}$ for $\ell < n-1$ implies strong convergence of the lower order terms $D^\ell A_i^\alpha \rightarrow D^\ell A_*^\alpha$ in $L^{2n/\ell+1}$. Having established the Euler-Lagrange system and the Coulomb condition (8.29) we can apply the Regularity Theorem 7.1 in order to deduce $A_*^\alpha \in C^\infty(U_\alpha)$ for every $\alpha = 1, \dots, L$. From the transformation rule (8.30) we then also infer

$\phi_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta)$ for $\alpha, \beta = 1, \dots, L$. From (8.30) and (8.31) we know that the local connection forms A_*^α stem from a smooth connection A_* on a new smooth principal bundle P_* that is given by the transition functions $\phi_{\alpha\beta}$.

Step 3: Identification of the limit bundle and conclusion. It remains to show that the bundle P_* , determined by the transition functions $\phi_{\alpha\beta}$, is isomorphic to the original bundle P with transition functions $g_{\alpha\beta}$. This step relies crucially on the subcritical dimension $m < 2n$, in particular on the compact embedding $W^{n,2} \hookrightarrow C^0$ that fails to hold in the critical dimension. In view of this embedding and the weak convergence (8.28)₂, we can assume that all $\phi_{\alpha\beta}^i$ are uniformly close to $\phi_{\alpha\beta}$ in the sense

$$\sup_{x \in U_\alpha \cap U_\beta} d(\phi_{\alpha\beta}^i(x), \phi_{\alpha\beta}(x)) \leq \frac{1}{2}\delta_M \quad \text{for all } i \in \mathbb{N}, \quad (8.32)$$

where δ_M denotes the constant from Lemma 8.5. Next, we observe that

$$\phi_{\alpha\beta}^1 = (g_\alpha^{-1}u_{1,\alpha})^{-1} g_{\alpha\beta} (g_\beta^{-1}u_{1,\beta}) \quad \text{on } U_\alpha \cap U_\beta.$$

The gauge transformations $g_\alpha^{-1}u_{1,\alpha}$ may not be smooth, but by the embedding $W^{n,2} \hookrightarrow C^0$ they are continuous. Therefore they can be approximated uniformly by smooth gauge transformations $v_\alpha \in C^\infty(U_\alpha)$. Letting $\tilde{g}_{\alpha\beta} := v_\alpha^{-1}g_{\alpha\beta}v_\beta$, we can thereby achieve

$$\sup_{x \in U_\alpha \cap U_\beta} d(\tilde{g}_{\alpha\beta}(x), \phi_{\alpha\beta}^1(x)) \leq \frac{1}{2}\delta_M.$$

Joining this with (8.32), we conclude

$$\sup_{x \in U_\alpha \cap U_\beta} d(\tilde{g}_{\alpha\beta}(x), \phi_{\alpha\beta}(x)) \leq \delta_M.$$

It is straightforward to check that the new transition functions $\tilde{g}_{\alpha\beta}$ still satisfy the cocycle conditions. We are therefore in a position to apply Lemma 8.5, which provides us with open sets $V_\alpha \subset U_\alpha$ that still cover M and gauge transformations $\phi_\alpha \in C^\infty(V_\alpha, G)$ with

$$\phi_\alpha^{-1} \phi_{\alpha\beta} \phi_\beta = \tilde{g}_{\alpha\beta} = v_\alpha^{-1} g_{\alpha\beta} v_\beta \quad \text{on } V_\alpha \cap V_\beta.$$

Abbreviating $\psi_\alpha := \phi_\alpha v_\alpha^{-1}$, we can re-write this to

$$\psi_\alpha^{-1} \phi_{\alpha\beta} \psi_\beta = g_{\alpha\beta} \quad \text{on } V_\alpha \cap V_\beta.$$

This means that P_* is isomorphic to P by the bundle isomorphism that is locally given by $\psi_\alpha := \phi_\alpha v_\alpha^{-1} \in C^\infty(V_\alpha, G)$. Letting $A^\alpha := \psi_\alpha^* A_*^\alpha$, we observe

$$g_{\alpha\beta}^* A^\alpha = (\psi_\alpha g_{\alpha\beta})^* A_*^\alpha = (\phi_{\alpha\beta} \psi_\beta)^* A_*^\alpha = A^\beta.$$

Therefore, the A^α are the localizations of a connection A on the bundle P . In order to check that this is the desired minimizer, we use the lower semicontinuity of the norm with respect to weak convergence for the estimate

$$Y_n(A) = Y_n(A_*) \leq \lim_{i \rightarrow \infty} Y_n(A_i) = \inf Y_n,$$

where the infimum is taken over all $W^{n-1,2}$ -connections on P . This shows the minimization property of A and concludes the proof of the theorem. \square

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